# A Companion to Classical Electrodynamics $3^{r d}$ Edition by J.D. Jackson 

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A lot of things can be said about Classical Electrodynamics, the third edition, by David J. Jackson. It's seemingly exhaustive, well researched, and certainly popular. Then, there is a general consensus among teachers that this book is the definitive graduate text on the subject. In my opinion, this is quite unfortunate. The text often assumes familiarity with the material, skips vital steps, and provides too few examples. It is simply not a good introductory text. On the other hand, Jackson was very ambitious. Aside from some notable omissions (such as conformal mapping methods), Jackson exposes the reader to most of classical electro-magnetic theory. Even Thomas Aquinas would be impressed! As a reference, Jackson's book is great!

It is obvious that Jackson knows his stuff, and in no place is this more apparent than in the problems which he asks at the end of each chapter. Sometimes the problems are quite simple or routine, other times difficult, and quite often there will be undaunting amounts of algebra required. Solving these problems is a time consuming endevour for even the quickest reckoners among us. I present this Companion to Jackson as a motivation to other students. These problems can be done! And it doesn't take Feynmann to do them.

Hopefully, with the help of this guide, lots of paper, and your own wits; you'll be able to wrestle with the concepts that challenged the greatest minds of the last century.

Before I begin, I will recommend several things which I found useful in solving these problems.

- Buy Griffiths' text, an Introduction to Electrodynamics. It's well written and introduces the basic concepts well. This text is at a more basic level than Jackson, and to be best prepared, you'll have to find other texts at Jackson's level. But remember Rome wasn't build in a day, and you have to start somewhere.
- Obtain other texts on the level (or near to it) of Jackson. I recommend Vanderlinde's Electromagnetism book or Eyges' Electromagnetism book. Both provide helpful insights into what Jackson is talking about. But even more usefully, different authors like to borrow each others' problems and examples. A problem in Jackson's text might be an example in one of these other texts. Or the problem might be rephrased in the other text; the rephrased versions often provide insight into what Jackson's asking! After all half the skill in writing a hard
physics problem is wording the problem vaguely enough so that no one can figure out what your talking about.
- First try to solve the problem without even reading the text. More often than not, you can solve the problem with just algebra or only a superficial knowledge of the topic. It's unfortunate, but a great deal of physics problems tend to be just turning the crank. Do remember to go back and actually read the text though. Solving physics problems is meaningless if you don't try to understand the basic science about what is going on.
- If you are allowed, compare your results and methods with other students. This is helpful. People are quick to tear apart weak arguments and thereby help you strengthen your own understanding of the physics. Also, if you are like me, you are a king of stupid algebraic mistakes. If ten people have one result, and you have another, there's a good likelihood that you made an algebraic mistake. Find it. If it's not there, try to find what the other people could have done wrong. Maybe, you are both correct!
- Check journal citations. When Jackson cites a journal, find the reference, and read it. Sometimes, the problem is solved in the reference, but always, the references provide vital insight into the science behind the equations.

A note about units, notation, and diction is in order. I prefer $S I$ units and will use these units whenever possible. However, in some cases, the use of Jacksonian units is inevitable, and I will switch without warning, but of course, I plan to maintain consistency within any particular problem. I will set $c=1$ and $\hbar=1$ when it makes life easier; hopefully, I will inform the reader when this happens. I have tried, but failed, to be regular with my symbols. In each case, the meaning of various letters should be obvious, or else if I remember, I will define new symbols. I try to avoid the clumsy $d^{3} \vec{x}$ symbols for volume elements and the $d^{2} \vec{x}$ symbols for area elements; instead, I use $d V$ and $d A$. Also, I will use $\hat{x}, \hat{y}$, and $\hat{z}$ instead of $\hat{i}, \hat{j}$, and $\hat{k}$. The only times I will use $i j k$ 's will be for indices.

Please, feel free to contact me, rmagyar@eden.rutgers.edu, about any typos or egregious errors. I'm sure there are quite a few.

Now, the fun begins...

## Problem 1.1

## Use Gauss' theorem to prove the following:

a. Any excess charge placed on a conductor must lie entirely on its surface.
In Jackson's own words, "A conductor by definition contains charges capable of moving freely under the action of applied electric fields". That implies that in the presence of electric fields, the charges in the conductor will be accelerated. In a steady configuration, we should expect the charges not to accelerate. For the charges to be non-accelerating, the electric field must vanish everywhere inside the conductor, $\vec{E}=0$. When $\vec{E}=0$ everywhere inside the conductor ${ }^{1}$, the divergence of $\vec{E}$ must vanish. By Gauss's law, we see that this also implies that the charge density inside the conductor vanishes: $0=\nabla \cdot \vec{E}=\frac{\rho}{\epsilon_{0}}$.
b. A closed, hollow conductor shields its interior from fields due to charges outside, but does not shield its exterior from the fields due to charges placed inside it.
The charge density within the conductor is zero, but the charges must be located somewhere! The only other place in on the surfaces. We use Gauss's law in its integral form to find the field outside the conductor.

$$
\int \vec{E} \cdot d \vec{A}=\frac{1}{\epsilon_{0}} \sum q_{i}
$$

Where the sum is over all enclosed charges. Evidently, the field outside the conductor depends on the surface charges and also those charges concealed deep within the cavities of the conductor.
c. The electric field at the surface of a conductor is normal to the surface and has a magnitude $\frac{\sigma}{\epsilon_{0}}$, where $\sigma$ is the charge density per unit area on the surface.
We assume that the surface charge is static. Then, $\vec{E}$ at the surface of a conductor must be normal to the surface; otherwise, the tangential components of the $E$-field would cause charges to flow on the surface, and that would contradict the static condition we already assumed. Consider a small area.

$$
\int \nabla \cdot \vec{E} d V=\int \vec{E} \cdot d \vec{A}=\int \frac{\rho}{\epsilon_{0}} d V
$$

[^0]But $\rho=0$ everywhere except on the surface so $\rho$ should more appropriately be written $\sigma \delta(f(\vec{x}))$. Where the function $f(\vec{x})$ subtends the surface in question. The last integral can then be written $\int \frac{\sigma}{\epsilon_{0}} \hat{n} \cdot d \vec{A}$. Our equation can be rearranged.

$$
\int \vec{E} \cdot d \vec{A}=\int \frac{\sigma}{\epsilon_{0}} \hat{n} \cdot d \vec{A} \rightarrow \int\left(\vec{E}-\frac{\sigma}{\epsilon_{0}} \hat{n}\right) \cdot d \vec{A}=0
$$

And we conclude

$$
\vec{E}=\frac{\sigma}{\epsilon_{0}} \hat{n}
$$

## Problem 1.3

Using Dirac delta functions in the appropriate coordinates, express the following charge distributions as three-dimensional charge densities $\rho(\vec{x})$.
a. In spherical coordinates, a charge $Q$ distributed over spherical shell of radius, $R$.
The charge density is zero except on a thin shell when $r$ equals $R$. The charge density will be of the form, $\rho \propto \delta(r-R)$. The delta function insures that the charge density vanishes everywhere except when $r=R$, the radius of the sphere. Integrating $\rho$ over that shell, we should get $Q$ for the total charge.

$$
\int A \delta(r-R) d V=Q
$$

$A$ is some constant yet to be determined. Evaluate the integral and solve for A.

$$
\int A \delta(r-R) d V=\int A \delta(r-R) r^{2} d(\cos \theta) d \phi d r=4 \pi R^{2} A=Q
$$

So $A=\frac{Q}{4 \pi R^{2}}$, and

$$
\rho(\vec{r})=\frac{Q}{4 \pi R^{2}} \delta(r-R)
$$

b. In cylindrical coordinates, a charge $\lambda$ per unit length uniformly distributed over a cylindrical surface of radius $b$.

$$
\int A \delta(r-b) d A=\lambda
$$

Since we are concerned with only the charge density per unit length in the axial direction, the integral is only over the plane perpendicular to the axis of the cylinder. Evaluate the integral and solve for A.

$$
\int B \delta(r-b) d A=\int B \delta(r-b) r d \theta d r=2 \pi b B=\lambda
$$

So $B=\frac{\lambda}{2 \pi b}$, and

$$
\rho(\vec{r})=\frac{\lambda}{2 \pi b} \delta(r-b)
$$

c. In cylindrical coordinates, a charge, $Q$, spread uniformly over a flat circular disc of negligible thickness and radius, $R$.

$$
\int A \Theta(r-R) \delta(z) d V=Q
$$

The $\Theta$ function of $x$ vanishes when $x$ is negative; when $x$ is positive, $\Theta$ is unity.

$$
\int A \Theta(R-r) \delta(z) d V=\int A \Theta(R-r) \delta(z) r d \theta d z d r=\pi R^{2} A=Q
$$

So $A=\frac{Q}{\pi R^{2}}$, and

$$
\rho(\vec{r})=\frac{Q}{\pi R^{2}} \Theta(R-r) \delta(z)
$$

d. The same as in part $c$, but using spherical coordinates.

$$
\int A \Theta(R-r) \delta\left(\theta-\frac{\pi}{2}\right) d V=Q
$$

Evaluate the integral and solve for A.

$$
\begin{array}{r}
\int A \Theta(R-r) \delta\left(\theta-\frac{\pi}{2}\right) d V=\int A \Theta(R-r) \delta\left(\theta-\frac{\pi}{2}\right) r^{2} d(\cos \theta) d \phi d r \\
=2 \pi R^{2} A=Q
\end{array}
$$

So $A=\frac{Q}{2 \pi R^{2}}$, and

$$
\rho(\vec{r})=\frac{Q}{2 \pi R^{2}} \Theta(R-r) \delta\left(\theta-\frac{\pi}{2}\right)
$$

## Problem 1.5

The time-averaged potential of a neutral hydrogen atom is given by

$$
\Phi=q \frac{e^{-\alpha r}}{r}\left(1+\frac{1}{2} \alpha r\right)
$$

where $q$ is the magnitude of the electronic charge, and $\alpha^{-1}=\frac{a_{0}}{2}$, $a_{0}$ being the Bohr radius. Find the distribution of charge (both continuous and discrete) that will give this potential and interpret your result physically.
We are given the time average potential for the Hydrogen atom.

$$
\Phi=q \frac{e^{-\alpha r}}{r}\left(1+\frac{1}{2} \alpha r\right)
$$

Since this potential falls off faster than $\frac{1}{r}$, it is reasonable to suspect that the total charge described by this potential is zero. If there were any excess charge ( + of - ) left over, it would have to produce a $\frac{1}{r}$ contribution to the potential.
Theoretically, we could just use Poisson's equation to find the charge density.

$$
\rho=-\epsilon_{0} \nabla^{2} \Phi=-\frac{\epsilon_{0}}{r^{2}} \frac{d}{d r}\left(r^{2} \frac{d \Phi}{d r}\right)
$$

But life just couldn't be that simple. We must be careful because of the singular behavior at $r=0$. Try $\Phi^{\prime}=-\frac{q}{r}+\Phi$. This trick amounts to adding a positive charge at the origin. We will have to subtract this positive charge from our charge distribution later.

$$
\Phi^{\prime}=q\left(\frac{e^{-\alpha r}-1}{r}\right)+\frac{1}{2} q \alpha e^{-\alpha r}
$$

which has no singularities. Plug into Poisson's equation to get

$$
\rho^{\prime}=-\frac{\epsilon_{0}}{r^{2}} \frac{d}{d r}\left(r^{2} \frac{d \Phi}{d r}\right)=-\frac{1}{2} \epsilon_{0} q \alpha^{3} e^{-\alpha r}
$$

The total charge density is then

$$
\rho(\vec{r})=\rho^{\prime}(\vec{r})+q \delta(\vec{r})=-\frac{1}{2} \epsilon_{0} q \alpha^{3} e^{-\alpha r}+q \delta(\vec{r})
$$

Obviously, the second terms corresponds to the positive nucleus while the first is the negative electron cloud.

## Problem 1.10

Prove the Mean Value Theorem: for charge free space the value of the electrostatic potential at any point is equal to the average of the potential over the surface of any sphere centered on that point. The average value of the potential over the spherical surface is

$$
\bar{\Phi}=\frac{1}{4 \pi R^{2}} \int \Phi d A
$$

If you imagine the surface of the sphere as discretized, you can rewrite the integral as an infinite sum: $\frac{1}{a} \int d A \rightarrow \sum_{\text {area }}$. Then, take the derivative of $\bar{\Phi}$ with respect to $R$.

$$
\frac{d \bar{\Phi}}{d R}=\frac{d}{d R} \sum \Phi=\sum \frac{d \Phi}{d R}
$$

You can move the derivative right through the sum because derivatives are linear operators. Convert the infinite sum back into an integral.

$$
\frac{d \bar{\Phi}}{d R}=\sum \frac{d \Phi}{d R}=\frac{1}{4 \pi R^{2}} \int \frac{d \Phi}{d R} d A
$$

One of the recurring themes of electrostatics is $\frac{d \Phi}{d R}=-E_{n}$. Use it.

$$
\frac{d \bar{\Phi}}{d R}=\frac{1}{4 \pi R^{2}} \int \frac{d \Phi}{d R} d A=-\frac{1}{4 \pi R^{2}} \int E_{n} d A=0
$$

By Gauss's law, $\int E_{n} d A=0$ since $q_{\text {included }}=0$. And so we have the mean value theorem:

$$
\frac{d \bar{\Phi}}{d R}=0 \rightarrow \bar{\Phi}_{\text {surface }}=\Phi_{\text {center }}
$$

q.e.d.

## Problem 1.12

Prove Green's Reciprocation Theorem: If $\Phi$ is the potential due to a volume charge density $\rho$ within a volume $V$ and a surface charge density $\sigma$ on the conducting surface $S$ bounding the volume $V$, while $\Phi^{\prime}$ is the potential due to another charge distribution $\rho^{\prime}$ and $\sigma^{\prime}$, then

$$
\int \rho \Phi^{\prime} d V+\int \sigma \Phi^{\prime} d A=\int \rho^{\prime} \Phi d V+\int \sigma^{\prime} \Phi d A
$$

Green gave us a handy relationship which is useful here. Namely,

$$
\int_{V}\left(\phi \nabla^{2} \psi-\psi \nabla^{2} \phi\right) d V^{\prime}=\oint_{S}\left[\phi \frac{\partial \psi}{\partial n}-\psi \frac{\partial \phi}{\partial n}\right] d A
$$

Let $\phi=\Phi$ and $\psi=\Phi^{\prime}$.

$$
\int_{V}\left(\Phi \nabla^{2} \Phi^{\prime}-\Phi^{\prime} \nabla^{2} \Phi\right) d V^{\prime}=\oint_{S}\left[\Phi \frac{\partial \Phi^{\prime}}{\partial n}-\Phi^{\prime} \frac{\partial \Phi}{\partial n}\right] d A
$$

Use Gauss's law, $\nabla^{2} \Phi=\frac{\rho}{\epsilon_{0}}$, to replace the Laplacian's on the left side of the equal sign with charge densities. From problem 1.1, we know $\frac{\partial \Phi}{\partial n}=\frac{\sigma}{\epsilon_{0}}$. Replace the derivatives on the right side by surface charge densities.

$$
\frac{1}{\epsilon_{0}} \int_{V}\left(\Phi \rho^{\prime}-\Phi^{\prime} \rho\right) d^{3} x^{\prime}=\frac{1}{\epsilon_{0}} \oint_{S}\left[\Phi \sigma^{\prime}-\Phi^{\prime} \sigma\right] d A
$$

With a tiny bit of rearrangement, we get Green's reciprocity theorem:

$$
\int \rho \Phi^{\prime} d V+\int \sigma \Phi^{\prime} d A=\int \rho^{\prime} \Phi d V+\int \sigma^{\prime} \Phi d A
$$



## Problem 1.13

Two infinite grounded conducting planes are separated by a distance $d$. A point charge $q$ is placed between the plans. Use the reciprocation theorem to prove that the total induced charge on one of the planes is equal to $(-q)$ times the fractional perpendicular distance of the point charge from the other plane.
Two infinite grounded parallel conducting planes are separated by a distance $d$. A charge, $q$, is placed between the plates.
We will be using the Green's reciprocity theorem

$$
\int \rho \Phi^{\prime} d V+\int \sigma \Phi^{\prime} d A=\int \rho^{\prime} \Phi d V+\int \sigma^{\prime} \Phi d A
$$

For the unprimed case, we have the situation at hand. $\rho$ and $\sigma$ vanish at all points except at the two plates' surfaces and at the point charge. The potential at the two grounded plates vanishes.
We need to choose another situation with the same surfaces for which we know the potential. The easiest thing that comes to mind is the parallel plate capacitor. We take the first plate to be at $x=0$ and the second at $x=d$. The charge density vanishes everywhere except on the two plates. The electrostatic potential is simple, $\Phi^{\prime}(x)=\Phi_{0} \frac{x}{d}$ which we know is true for the parallel plate capacitor.
Plugging into Green's reciprocity theorem, we have

$$
\left(q \times \Phi_{0} \frac{x}{d}\right)+\left(0+q^{\prime} \Phi_{0} \frac{d}{d}\right)=(0)+(0)
$$

With a little algebra, this becomes

$$
q^{\prime}=-\frac{x}{d} q
$$

on plate two. By symmetry, we can read off the induced charge on the other plate, $q^{\prime}=-\frac{d-x}{d} q=-\left(1-\frac{x}{d}\right) q$.

## Bonus Section: A Clever Ruse

This tricky little problem was on my qualifying exam, and I got it wrong. The irony is that I was assigned a similar question as an undergrad. I got it wrong back then, thought, "Whew, I'll never have to deal with this again," and never looked at the solution. This was a most foolish move.
Calculate the force required to hold two hemispheres (radius $R$ ) each with charge $Q / 2$ together.
Think about a gaussian surface as wrapping paper which covers both hemispheres of the split orb. Now, pretend one of the hemispheres is not there. Since Gauss's law only cares about how much charge is enclosed, the radial field caused by one hemisphere is

$$
\vec{E}=\frac{1}{2} \frac{1}{4 \pi \epsilon_{0}} \frac{Q}{R^{2}} \hat{r}
$$

Because of cylindrical symmetry, we expect the force driving the hemispheres apart to be directed along the polar axis. The non polar components cancel, so we need to consider only the polar projection of the electric field. The assumption is that we can find the polar components of the electric field by taking $z$ part of the radial components. So we will find the northwardly directed electric field created by the southern hemisphere and affecting the northern hemisphere and integrate this over the infinitesimal charge elements of the northern hemisphere. Using $d q=\frac{Q}{4 \pi R^{2}} d A$, we have

$$
F_{z}=\int_{\text {north }} E_{z} d q=\int\left(\frac{1}{4 \pi \epsilon_{0}} \frac{Q}{2 R^{2}} \cos \theta\right) \frac{Q}{4 \pi R^{2}} d A
$$

where $\theta$ is the angle the electric field makes with the $z$-axis.

$$
F_{z}=-\frac{1}{4 \pi \epsilon_{0}} \frac{Q^{2}}{8 \pi R^{4}} 2 \pi R^{2} \int_{0}^{1} \cos \theta d(\cos \theta)=-\frac{Q^{2}}{32 \pi \epsilon_{0} R^{2}}
$$

The conclusions is that we have to push down on the upper hemisphere if the bottom is fixed, and we want both shells to stay together.

## Problem 2.1

A point charge $q$ is brought to a position a distance $d$ away from an infinite plane conductor held at zero potential. Using the method of images, find:
a. the surface charge density induced on the plane, and plot it; Jackson asks us to use the method of images to find the potential for a point charge placed a distance, $d$, from a infinitely large zero potential conducting $x-z$ sheet located at $y=0$.

$$
\Phi(\vec{r})=\frac{\frac{1}{4 \pi \epsilon_{0}} q}{\left|\overrightarrow{r_{o}}-\overrightarrow{r_{q}^{\prime}}\right|}+\frac{\frac{1}{4 \pi \epsilon_{0}} q_{I}}{\left|\overrightarrow{r_{o}}-\overrightarrow{r_{I}^{\prime}}\right|}
$$

The first term is the potential contribution from the actual charge $q$ and the second term is the contribution from the image charge $q_{I}$. Let the coordinates $x, y$, and $z$ denote the position of the field in question, while the coordinates $x_{0}, y_{0}$, and $z_{0}$ denote the position of the actual charge. Choose a coordinate system so that the real point charge is placed on the positive $y$-axis. $x_{0}$ and $y_{0}$ vanish in this coordinate system. Now, apply boundary conditions $\Phi(y=0)=0$.

$$
\Phi(y=0)=\frac{\frac{1}{4 \pi \epsilon_{0}} q}{\sqrt{x^{2}+z^{2}+y_{0}^{2}}}+\frac{\frac{1}{4 \pi \epsilon_{0}} q^{\prime}}{\sqrt{\left(x-x_{I}^{\prime}\right)^{2}+y-y_{I}^{\prime 2}+\left(z-z_{I}^{\prime}\right)^{2}}}=0
$$

We can have $\Phi=0$ for all points on the $x-z$ plane only if $q^{\prime}=-q, x_{I}^{\prime}=0$, $z_{I}^{\prime}=0$, and $y_{I}^{\prime}=-y_{0}$. Label $y_{0}=d$.

$$
\Phi(x, y, z)=\frac{1}{4 \pi \epsilon_{0}} q\left(\frac{1}{\sqrt{x^{2}+(y-d)^{2}+z^{2}}}-\frac{1}{\sqrt{x^{2}+(y+d)^{2}+z^{2}}}\right)
$$

To find the surface charge density induced on the sheet, we use the formula from problem 1.1.

$$
\begin{gathered}
\sigma=\epsilon_{0} E_{n}=-\left.\epsilon_{0} \frac{\partial \Phi}{\partial y}\right|_{y=0} \\
\sigma=-\frac{1}{4 \pi \epsilon_{0}} q \frac{2 d}{\left(x^{2}+d^{2}+z^{2}\right)^{\frac{3}{2}}}=-\frac{q}{4 \pi \epsilon_{0} d^{2}}\left(\frac{2}{\left(1+x^{2} / d^{2}+y^{2} / d^{2}\right)^{\frac{3}{2}}}\right)
\end{gathered}
$$


b. the force between the plane and the charge by using Coulumb's law for the force between the charge and its image;
The force between the charge and its image is given by Coulumb's law.

$$
\vec{F}=\frac{1}{4 \pi \epsilon_{0}} \frac{q q^{\prime}}{\left|\overrightarrow{r_{q}}-\overrightarrow{r_{I}^{\prime}}\right|^{\prime}} \hat{y}=-\frac{1}{4 \pi \epsilon_{0}} \frac{q^{2}}{4 d^{2}} \hat{y}
$$

Where the effective distance between the charge and image is $\left|\overrightarrow{r_{q}^{\prime}}-\overrightarrow{r_{I}^{\prime}}\right|=2 d$. The force is obviously attractive because of the minus sign.
c. the total force acting on the plane by integrating $\frac{\sigma^{2}}{2 \epsilon_{0}}$ over the whole plane;
Now, we use the method Jackson suggests. First, we square our equation for $\sigma$.

$$
\sigma^{2}=\frac{q^{2}}{16 \pi^{2}} \frac{4 d^{2}}{\left(x^{2}+d^{2}+z^{2}\right)^{3}}
$$

Jackson tell us that the force can be computed from the following integral:

$$
\vec{F}=\int \frac{\sigma^{2}}{2 \epsilon_{0}} d \vec{A}
$$

So we do this integral.

$$
\vec{F}=\int_{0}^{\infty} \int_{0}^{2 \pi} \frac{q^{2}}{32 \pi^{2} \epsilon_{0}} \frac{r d^{2}}{\left(r^{2}+d^{2}\right)^{3}} d \theta d r \hat{y}
$$

where $r^{2}=x^{2}+z^{2}$. Let $u=r^{2}+d^{2}$ and $d u=2 r d r$.

$$
\vec{F}=\int_{-d^{2}}^{\infty} \frac{q^{2}}{16 \pi \epsilon_{0}} \frac{1}{2} \frac{d^{2}}{u^{3}} d u \hat{y}=-\frac{1}{4 \pi \epsilon_{0}} \frac{q^{2}}{4 d^{2}} \hat{r}
$$

Which is the same as in part b.
d. The work necessary to remove the charge $q$ from its position at $d$ to infinity;

$$
W=\int \mathbf{F} \cdot \mathbf{r}=-\int_{d}^{\infty} \frac{1}{4 \pi \epsilon_{0}} \frac{q^{2}}{4 r^{2}}=\left.\frac{1}{4 \pi \epsilon_{0}} \frac{q^{2}}{4 r}\right|_{d} ^{\infty}=-\frac{q^{2}}{16 \pi \epsilon_{0} d}
$$

The image charge is allowed to move in the calculation.
e. The potential energy between the charge $q$ and its image. Compare to part d.

$$
U=-\frac{1}{2} \frac{1}{4 \pi \epsilon_{0}} \frac{q^{2}}{\left|r-r^{\prime}\right|}=-\frac{q^{2}}{8 \pi \epsilon_{0} d}
$$

Here we find the energy without moving the image charge so our result is different from part d.
f. Find the answer to part d in electron volts for an electron originally one Angstrom from the surface.
Use the result from part d. Take $d \approx 1$ Angstrom so $W=\frac{1}{4 \pi \epsilon_{0}} \frac{q^{2}}{4 d}=5.77 \times$ $10^{-19}$ joules or 3.6 eV .

## Problem 2.2

Using the method of images, discuss the problem of a point charge $q$ inside a hollow, grounded, conducting sphere of inner radius $a$. Find. . .
I botched this one up the first time I did it. Hopefully, this time things will turn out better!
a. the potential inside the sphere

As implied by definition of conducting $V=0$ on the surface. We must place an image charge outside the sphere on the axis defined by the real charge $q$ and the center of the sphere. Use a Cartesian coordinate system and set the $x$-axis to be the axis defined by the charge, its image, and the center of the sphere.

$$
\Phi=\frac{1}{4 \pi \epsilon_{0}}\left[\frac{q}{\sqrt{\left(x-x_{1}\right)^{2}+y^{2}+z^{2}}}+\frac{q^{\prime}}{\sqrt{\left(x-x_{2}^{\prime}\right)^{2}+y^{2}+z^{2}}}\right]
$$

The charge $q$ is positioned at $x_{1}$ and its image $q^{\prime}$ is at $x_{2}^{\prime}{ }^{2}$. For the real charge outside the sphere and its image inside, Jackson finds $q_{\text {in }}=-\frac{a}{x_{\text {out }}} q_{\text {out }}$ and $x_{\text {in }}=\frac{a^{2}}{x_{\text {out }}}$. We let $x_{\text {in }}=x_{1}$ and $x_{\text {out }}=x_{2}^{\prime}$, and the second equations tells us: $x_{2}^{\prime}=\frac{a^{2}}{x_{1}}$. Let $q_{i n}=q$ and $q_{o u t}=q^{\prime}$. Care must be taken because the first equation depends on $x_{o} u t=x_{2} . q=-\frac{a}{x_{2}} q^{\prime}=-\frac{x_{1}}{a} q^{\prime}$. So $q^{\prime}=-\frac{a}{x_{1}} q$. Incidentally, even if I had no help from Jackson's text, this is a good guess because dimensionally it works. This image charge distribution does satisfy the boundary conditions.

$$
\Phi(a)=\frac{1}{4 \pi \epsilon_{0}} q\left[\frac{1}{\sqrt{x_{1}^{2}+a^{2}}}-\frac{a}{x_{1}} \frac{1}{\sqrt{\left(\frac{a^{2}}{x_{1}}\right)^{2}+a^{2}}}\right]=0
$$

A more rigorous determination in not necessary because this function is unique. Therefore, for a real charge $q$ placed within a conducting sphere of radius $a$, we find the potential to be:

$$
\Phi(x, y, z)=\frac{1}{4 \pi \epsilon_{0}} q\left[\frac{1}{\sqrt{\left(x-x_{1}\right)^{2}+y^{2}+z^{2}}}-\frac{a}{x_{1}} \frac{1}{\sqrt{\left(x-\frac{a^{2}}{x_{1}}\right)^{2}+y^{2}+z^{2}}}\right]
$$

[^1]where $x_{1}<a$ for the charge inside the sphere and $x_{1} \neq 0$. The charge should not be placed at the center of the sphere. I am sure that a limiting method could reveal the potential for a charge at the center, but that is not necessary. Use Gauss's law to get
$$
\Phi=\frac{1}{4 \pi \epsilon_{0}} \frac{q}{r}-\frac{1}{4 \pi \epsilon_{0}} \frac{q}{a}
$$

## b. the induced surface charge density

The surface charge density will simply be the same as calculated by Jackson for the inverse problem. For a charge outside a conducting sphere, the surface charge density is such.

$$
\sigma=-\frac{1}{4 \pi \epsilon_{0}} q \frac{a}{x_{1}} \frac{1-\frac{a^{2}}{x_{1}^{2}}}{\left(1+\frac{a^{2}}{x_{1}^{2}}-2 \frac{a}{x_{1}} \cos \gamma\right)^{\frac{3}{2}}}
$$

where $\gamma$ is the angle between the $x$-axis and the area element. Jackson's result comes from taking $\sigma=-\epsilon_{0} \frac{\partial \Phi}{\partial n}$, but our potential is functionally the same. Thus, our surface charge distribution will be the same.

$$
\sigma=-\frac{1}{4 \pi \epsilon_{0}} q \frac{a}{x_{1}} \frac{1-\frac{a^{2}}{x_{1}^{2}}}{\left(1+\frac{a^{2}}{x_{1}^{2}}-2 \frac{a}{x_{1}} \cos \gamma\right)^{\frac{3}{2}}}
$$

c. the magnitude and direction of the force acting on $q$.

The force acting on $q$ can be obtained by Coulomb's law.

$$
F=\frac{1}{4 \pi \epsilon_{0}} \frac{q q^{\prime}}{\left|r-r^{\prime}\right|^{2}}=\frac{1}{4 \pi \epsilon_{0}} q\left(-\frac{a}{x_{1}} q\right) \frac{1}{\left(\frac{a^{2}}{x_{1}}-x_{1}\right)^{2}}=-\frac{1}{4 \pi \epsilon_{0}} q^{2} \frac{a x_{1}}{\left(a^{2}-x_{1}^{2}\right)^{2}}
$$

d. Is there any change in the solution if the sphere is kept at fixed potential $\Phi$ ? Is the sphere has a total charge $Q$ on its inner and outer surfaces?
If the sphere is kept at a fixed potential $\Phi$, we must add an image charge at the origin so that the potential at $R$ is $\Phi$. If the sphere has a total charge $Q$ on its inner and outer surfaces, we figure out what image charge would create a surface charge equal to $Q$ and place this image at the origin.

### 2.28

A closed volume is bounded by conducting surfaces that are the $n$ sides of a regular polyhedron $(n=4,6,8,12,20)$. The $n$ surfaces are at different potentials $\Phi_{i}, i=1,2, \ldots, n$. Prove in the simplest way you can that the potential at the center of the polyhedron is the average of the potential on the $n$ sides.
I will do a simple derivation. We have some crazy $n$-sided regular polyhedron. That means that each side has the same area and each corner has the same set of angles. If one side is at potential $\Phi_{i}$ but all the other sides are at zero potential. The potential in the center of the polygon will be some value, call it $\Phi_{i}^{\prime}$. By symmetry, we could use this same approach for any side; A potential $\Phi_{i}$ always produces another potential $\Phi_{i}^{\prime}$ at the center. Now, we use linear superposition. Let all the sides be at $\Phi_{i}$. Then, the potential at the center is

$$
\Phi_{\text {center }}=\sum_{i=1}^{n} \Phi_{i}^{\prime}
$$

If all the $\Phi_{i}$ are equal, then so are all the $\Phi_{i}^{\prime}$. Then, $\Phi_{c}=n \Phi_{i}^{\prime}$, and we can solve for $\Phi_{i}^{\prime}=\frac{\Phi_{c}}{n}$. If each surface is at some potential, $\Phi_{i}$, then the entire interior is at that potential, and $\Phi_{i}=\Phi_{c}$ according to the mean value theorem. Therefore, $\Phi_{i}^{\prime}=\frac{\Phi_{i}}{n}$ is the contribution from each side.
For a set of arbitrary potentials for each side, we can use the principle of linear superposition again.

$$
\Phi_{c}=\frac{1}{n} \sum_{i=1}^{n} \Phi_{i}
$$

q.e.d.

## Problem 3.3

A think, flat, circular, conducting disc of radius $R$ is located in the $x-y$ plane with its center at the origin and is maintained at a fixed potential $\Phi$. With the information that the charge density of the disc at fixed potential is proportional to $\left(R^{2}-\rho^{2}\right)^{-\frac{1}{2}}$, where $\rho$ is the distance out from the center of the disk. . .
Note $\rho$ is used where I usually use $r^{\prime}$.
a. Find the potential for $r>R$.

For a charged ring at $z=0$ on the $r-\phi$ plane, Jackson derived the following:

$$
\Phi(r, \theta)= \begin{cases}q \sum_{L=0}^{\infty} \frac{\rho^{L}}{L^{L+1}} P_{L}(0) P_{L}(\cos \theta), & r \geq R \\ q \sum_{L=0}^{\infty} \frac{r^{L}}{\rho^{L+1}} P_{L}(0) P_{L}(\cos \theta), r<R\end{cases}
$$

But

$$
P_{L}(0)=\left\{\begin{array}{l}
0, \text { for } L \text { odd } \\
(-1)^{\frac{L}{2}} \frac{(L+1)!!}{(L+1) L!!}=f(L), \text { for } L \text { even }
\end{array}\right.
$$

We can replace $L$ by $2 \ell$ because every other term vanishes.
Since $\sigma \propto\left(R^{2}-\rho^{2}\right)^{-\frac{1}{2}}$ on the disk, the total charge on the disk is

$$
Q=\int_{0}^{R} \frac{2 \pi \kappa \rho}{\sqrt{R^{2}-\rho^{2}}} d \rho
$$

Let $u=R^{2}-\rho^{2}, d u=-2 \rho d \rho$, so

$$
Q=-\int_{R^{2}}^{0} \frac{\pi \kappa}{\sqrt{u}} d u=\left.2 \pi \kappa u^{\frac{1}{2}}\right|_{0} ^{R^{2}}=2 \pi \kappa R
$$

And $\kappa=\frac{Q}{2 \pi R}$. Now, we solve for a disk made up of infinitely many infinitesimally small rings. Each contributes to the potential

$$
\delta \Phi(r, \theta)=\sigma \sum_{\ell=0}^{\ell} \frac{\rho^{2 \ell}}{r^{2 \ell+1}} f(2 \ell) P_{2 \ell}(\cos \theta) d A, \quad r \geq R
$$

where $f(2 \ell)=P_{2 \ell}(0)$. And integrating over the disk gives the total potential.

$$
\begin{aligned}
\Phi(r, \theta) & =\int \kappa\left(R^{2}-\rho^{2}\right)^{-\frac{1}{2}} \sum_{\ell=0}^{\ell} \frac{\rho^{2 \ell}}{r^{2 \ell+1}} f(2 \ell) P_{2 \ell}(\cos \theta) \rho d \rho d \phi \\
& =2 \pi \kappa \sum \int_{0}^{R}\left(R^{2}-\rho^{2}\right)^{-\frac{1}{2}} \frac{\rho^{2 \ell}}{r^{2 \ell+1}} f(2 \ell) P_{2 \ell}(\cos \theta) \rho d \rho
\end{aligned}
$$

Consider the integral over $\rho$.

$$
\int_{0}^{R} \frac{\rho^{2 \ell+1}}{\sqrt{R^{2}-\rho^{2}}} d \rho=\frac{1}{R} \int_{0}^{R} \frac{\rho^{2 \ell+1}}{\sqrt{1-\frac{\rho^{2}}{R^{2}}}} d \rho
$$

Let $\frac{\rho}{R}=\sin \theta, d \rho=R \cos \theta d \theta$.

$$
I_{1}=\frac{1}{R} \int_{0}^{\frac{\pi}{2}} R \frac{(R)^{2 \ell+1} \sin ^{2 \ell+1} \theta}{\cos \theta} \cos \theta d \theta=R^{2 \ell+1} \frac{2^{\ell} \ell!}{(2 \ell+1)!!}
$$

Using

$$
\int_{0}^{\frac{\pi}{2}} \sin ^{2 \ell+1} \theta d \theta=\frac{2^{\ell} \ell!}{(2 \ell+1)!!}
$$

So

$$
\Phi=2 \pi \kappa \sum \frac{2^{\ell} \ell!}{(2 \ell+1)!!} f(2 \ell) \frac{R^{2 \ell+1}}{r^{2 \ell+1}} P_{2 \ell}(\cos \theta)
$$

but we know $f(2 \ell)$.

$$
\Phi=\frac{4 Q}{R} \sum(-1)^{\ell} \frac{(2 \ell+1)!!}{(2 \ell+1)(2 \ell)!!} \frac{2^{\ell} \ell!}{(2 \ell+1)!!}\left(\frac{R}{r}\right)^{2 \ell}\left(\frac{R}{r}\right) P_{2 \ell}(\cos \theta)
$$

Since $(2 \ell)!!=2^{\ell} \ell!$,

$$
\Phi=\frac{4 Q}{R} \sum(-1)^{\ell} \frac{1}{2 \ell+1}\left(\frac{R}{r}\right)^{2 \ell}\left(\frac{R}{r}\right) P_{2 \ell}(\cos \theta), r \geq R
$$

The potential on the disk at the origin is V .

$$
V=\int_{0}^{2 \pi} \int_{0}^{R} \sigma \rho d \rho d \phi=\int \frac{2 Q}{\pi R} \frac{2 \pi \rho}{|\rho| \sqrt{R^{2}-\rho^{2}}} d \rho
$$

Using $\int \frac{d x}{\sqrt{a^{2}-x^{2}}}=\sin ^{-1}\left(\frac{x}{|a|}\right)$,

$$
V=\left.\frac{2 Q}{\pi R} 2 \pi \sin ^{-1}\left(\frac{x}{|R|}\right)\right|_{0} ^{R}=\frac{2 Q \pi}{R}
$$

And $\kappa=\frac{2 Q}{\pi R}=\frac{V}{\pi^{2}}$. Then,

$$
\Phi=\frac{2 V}{\pi}\left(\frac{R}{r}\right) \sum(-1)^{\ell} \frac{1}{2 \ell+1}\left(\frac{R}{r}\right)^{\ell} P_{2 \ell}(\cos \theta), r \geq R
$$

A similar integration can be carried out for $r<R$.

$$
\Phi=\frac{2 \pi Q}{R}-\frac{4 Q}{R} \sum(-1)^{\ell} \frac{1}{2 \ell+1}\left(\frac{r}{R}\right)^{2 \ell}\left(\frac{r}{R}\right) P_{2 \ell}(\cos \theta), r \leq R
$$

b. Find the potential for $r<R$.

I can't figure out what I did here. I'll get back to this.
c. What is the capacitance of the disc?
$C=\frac{Q}{V}$, but from part a $Q=\frac{2 V R}{\pi}$ so

$$
C=\frac{2 V R}{\pi}\left(\frac{1}{V}\right)=\frac{2 R}{\pi}
$$

## Problem 3.9

A hollow right circular cylinder of radius $b$ has its axis coincident with the $z$ axis and its ends at $z=0$ and $z=L$. The potential on the end faces is zero while the potential on the cylindrical surface is given by $\Phi(\phi, z)$. Using the appropriate separation of variables in cylindrical coordinates, find a series solution for the potential everywhere inside the cylinder.
$V=0$ at $z=0, L$. Because of cylindrical symmetry, we will try cylindrical coordinates. Then, we have

$$
\nabla^{2} \Phi=0 \rightarrow \frac{1}{r} \frac{\partial}{\partial r}\left(\frac{1}{r} \frac{\partial \Phi}{\partial r}\right)+\frac{1}{r^{2}} \frac{\partial^{2} \Phi}{\partial \phi^{2}}+\frac{\partial^{2} \Phi}{\partial z^{2}}=0
$$

Try $\Phi(r, \phi, z)=R(r) Z(z) Q(\phi)$. Separating the Laplace equation in cylindrical coordinates, we find three differential equations which must be satisfied.

$$
\frac{\partial^{2} Z}{\partial z^{2}}-k^{2} Z=0
$$

has the solution

$$
Z=A \sin (k z)+B \cos (k z)
$$

The solution must satisfy boundary conditions that $\Phi=0$ at $z=0, L$. Therefore, $B$ must vanish.

$$
Z=A \sin (k z)
$$

where $k=\frac{n \pi}{L}$.
Similarly, we have for $Q$

$$
\frac{\partial^{2} Q}{\partial \phi^{2}}-m^{2} Q=0
$$

which has the solution

$$
Q=C \sin (m \phi)+D \cos (m \phi)
$$

$m$ must be an integer for $Q$ to be single valued.
The radial part must satisfy the frightening equation. Note the signs. This is not the typical Bessel equations, but have no fear.

$$
\frac{\partial^{2} R}{\partial x^{2}}+\frac{1}{x} \frac{\partial R}{\partial x}-\left(1+\frac{m^{2}}{x^{2}}\right) R=0
$$

where $x=k r$. The solutions are just modified Bessel functions.

$$
R(x)=E I_{m}(x)+F K_{m}(x)
$$

$m$ must be an integer for $R$ to be single valued. $I_{m}$ and $K_{m}$ are related to other Bessel and Neumann functions via

$$
\begin{gathered}
I_{m}(k r)=i^{-m} J_{m}(i k r) \\
K_{m}(k r)=\frac{\pi}{2} i^{m+1} H_{m}^{(1)}(i k r)
\end{gathered}
$$

The potential is finite at $r=0$ so

$$
H_{m}^{(1)}(0)=J_{m}(0)+i N_{m}(0)=0
$$

But $K_{m} \neq 0$ so $F=0$.
We can now write $\Phi$ in a general form.

$$
\Phi=R Z Q=\sum A \sin \left(\frac{n \pi}{L} z\right)(C \sin (m \phi)+D \cos (m \phi)) E I_{m}\left(\frac{n \pi}{L} r\right)
$$

Let $A$ and $E$ be absorbed into $C$ and $D$.

$$
\Phi=\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sin \left(\frac{n \pi}{L} z\right) I_{m}\left(\frac{n \pi}{L} r\right)\left(C_{m n} \sin (m \phi)+D_{m n} \cos (m \phi)\right)
$$

Now, we match boundary conditions. At $r=b, \Phi(\phi, z)=V(\phi, z)$. So

$$
\Phi(\phi, z)=\sum_{m, n} \sin \left(\frac{n \pi}{L} z\right) I_{m}\left(\frac{n \pi}{L} b\right)\left(C_{m n} \sin (m \phi)+D_{m n} \cos (m \phi)\right)=V
$$

The $I_{m}\left(\frac{n \pi}{L} b\right)$ are just a set of constants so we'll absorb them into $C_{m n}^{\prime}$ and $D_{m n}^{\prime}$ for the time being. The coefficients, $C_{m n}^{\prime}$ and $D_{m n}^{\prime}$, can be obtained via Fourier analysis.

$$
\begin{aligned}
C_{m n}^{\prime} & =\kappa \int_{0}^{2 L} \int_{0}^{2 \pi} \Phi(\phi, z) \sin \left(\frac{n \pi}{L} z\right) \sin (m \phi) d \phi d z \\
D_{m n}^{\prime} & =\kappa \int_{0}^{2 L} \int_{0}^{2 \pi} \Phi(\phi, z) \sin \left(\frac{n \pi}{L} z\right) \cos (m \phi) d \phi d z
\end{aligned}
$$

$\kappa$ is determined by orthonormality of the various terms.

$$
\begin{aligned}
& \kappa^{-1}=\int_{0}^{2 L} \sin ^{2}\left(\frac{n \pi}{L} z\right) d z \int_{0}^{2 \pi} \sin ^{2}(m \phi) d \phi= \\
& \left.\frac{L}{\pi n m}\left[\frac{x}{2}-\frac{\sin (2 x)}{4}\right]\right|_{x=0} ^{2 \pi n} \times\left.\left[\frac{x}{2}-\frac{\sin (2 x)}{4}\right]\right|_{x=0} ^{2 \pi m}
\end{aligned}
$$

So $\kappa=\frac{1}{L \pi}$. Finally, we have

$$
C_{m n}=\frac{1}{L \pi} \frac{1}{I_{m}\left(\frac{n \pi}{L} b\right)} \int_{0}^{2 L} \int_{0}^{2 \pi} \Phi(\phi, z) \sin \left(\frac{n \pi}{L} z\right) \sin (m \phi) d \phi d z
$$

And

$$
D_{m n}=\frac{1}{L \pi} \frac{1}{I_{m}\left(\frac{n \pi}{L} b\right)} \int_{0}^{2 L} \int_{0}^{2 \pi} \Phi(\phi, z) \sin \left(\frac{n \pi}{L} z\right) \cos (m \phi) d \phi d z
$$

## Problem 3.14

A line charge of length $2 d$ with a total charge of $Q$ has a linear charge density varying as $\left(d^{2}-z^{2}\right)$, where $z$ is the distance from the midpoint. A grounded, conducting, spherical shell of inner radius $b>d$ is centered at the midpoint of the line charge.
a. Find the potential everywhere inside the spherical shell as an expansion in Legendre polynomials.

$$
Q=\int_{-d}^{d} \kappa\left(d^{2}-z^{2}\right) d z=\frac{4}{3} \kappa d^{3}
$$

so $\kappa=\frac{3 Q}{4 d^{3}}$ and

$$
\lambda=\frac{3 Q}{4 d^{3}}\left(d^{2}-z^{2}\right)
$$

For use later, we will write this in spherical coordinates.

$$
\rho(r, \theta, \phi)=\frac{3 Q}{4 d^{3}}\left(d^{2}-r^{2}\right) \frac{1}{\pi r^{2}} \delta\left(\cos ^{2} \theta-1\right)
$$

For the inside of the spherical shell, the Green's function is:

$$
G\left(x, x^{\prime}\right)=4 \pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{Y_{\ell m}\left(\theta^{\prime}, \phi^{\prime}\right) Y_{\ell m}(\theta, \phi)}{(2 \ell+1)\left[1-\left(\frac{a}{b}\right)^{2 \ell+1}\right]}\left(r_{<}^{\ell}-\frac{a^{2 \ell+1}}{r_{<}^{\ell+1}}\right)\left(\frac{1}{r_{>}^{\ell+1}}-\frac{r_{>}^{\ell}}{b^{2 \ell+1}}\right)
$$

Where $a$ and $b$ denote the inner and outer radii. Here $a=0$ so

$$
G\left(x, x^{\prime}\right)=4 \pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{Y_{\ell m}\left(\theta^{\prime}, \phi^{\prime}\right) Y_{\ell m}(\theta, \phi)}{(2 \ell+1)} r_{<}^{\ell}\left(\frac{1}{r_{>}^{\ell+1}}-\frac{r_{>}^{\ell}}{b^{2 \ell+1}}\right)
$$

because of azimuthal symmetry on $m=0$ terms contribute.

$$
G\left(x, x^{\prime}\right)=4 \pi \sum_{\ell=0}^{\infty} P_{\ell}\left(\cos \theta^{\prime}\right) P_{\ell}(\cos \theta) r_{<}^{\ell}\left(\frac{1}{r_{>}^{\ell+1}}-\frac{r_{>}^{\ell}}{b^{2 \ell+1}}\right)
$$

The potential can be obtained through Green's functions techniques.

$$
\Phi(x)=\frac{1}{4 \pi \epsilon_{0}} \int d^{3} x^{\prime} \rho\left(x^{\prime}\right) G\left(x, x^{\prime}\right)
$$

And explicitly
$\Phi=\frac{1}{4 \pi \epsilon_{0}} \int d \phi^{\prime} d\left(\cos \theta^{\prime}\right) r^{\prime 2} d r^{\prime} \frac{3 Q}{d^{3} r^{2}}\left(d^{2}-r^{\prime 2}\right) \delta\left(\cos ^{2} \theta^{\prime}-1\right) \sum_{\ell=0}^{\infty} P_{\ell}\left(\cos \theta^{\prime}\right) P_{\ell}(\cos \theta) r_{<}^{\ell}\left(\frac{1}{r_{>}^{\ell+1}}-\frac{r_{>}^{\ell}}{b^{2 \ell+1}}\right)$
The integrations over $\phi^{\prime}$ and $\theta^{\prime}$ are easy and fun!
$\Phi=\frac{3 Q}{16 \pi \epsilon_{0} d^{3}} \sum_{\ell=0}^{\infty}\left(P_{\ell}(1)+P_{\ell}(-1)\right) P_{\ell}(\cos \theta) \int_{0}^{b} d r^{\prime} r^{\prime 2}\left(d^{2}-r^{\prime 2}\right) r_{<}^{\ell}\left(\frac{1}{r_{>}^{\ell+1}}-\frac{r_{>}^{\ell}}{b^{2 \ell+1}}\right)$
Integration over $r^{\prime}$ must be done over several regions: $r<d$ and $r^{\prime}>r, r<d$ and $r^{\prime}<r, r>d$ and $r^{\prime}>r$, and $r>d$ and $r^{\prime}<r$. When the smoke clears, we find:

$$
\Phi(r, \theta, \phi)=\frac{3 Q}{16 \pi \epsilon_{0} d^{3}} \sum_{\ell=0}^{\infty}\left(P_{\ell}(1)+P_{\ell}(-1)\right) P_{\ell}(\cos \theta) \mathcal{I}(r, \ell)
$$

where

$$
\begin{array}{r}
\mathcal{I}(r, \ell)=\left(\frac{1}{r^{\ell+1}}-\frac{r^{\ell}}{b^{2 \ell+1}}\right)\left(d^{2} \frac{r^{\ell+1}}{\ell+1}-\frac{r^{\ell+3}}{\ell+3}\right) \\
+r^{\ell}\left[-\frac{d^{2-\ell}}{\ell}+\frac{d^{2-\ell}}{\ell-2}-\frac{d^{\ell+3}}{(\ell+1) b^{2 \ell+1}}-\frac{d^{\ell+3}}{(\ell+3) b^{2 \ell+1}}\right] \\
-r^{\ell}\left[-\frac{d^{2}}{\ell r^{\ell}}+\frac{r^{2-\ell}}{\ell-2}-\frac{d^{2} r^{\ell+1}}{(\ell+1) b^{2 \ell+1}}-\frac{r^{\ell+3}}{(\ell+3) b^{2 \ell+1}}\right]
\end{array}
$$

Presumably, this can be reduced, but I never got around to that. For $r<d$ and
$\Phi(r, \theta, \phi)=\frac{3 Q}{16 \pi \epsilon_{0} d^{3}} \sum_{\ell=0}^{\infty}\left(P_{\ell}(1)+P_{\ell}(-1)\right) P_{\ell}(\cos \theta)\left(\frac{1}{r^{\ell+1}}-\frac{r^{\ell}}{b^{2 \ell+1}}\right)\left(\frac{2 d^{\ell+3}}{(\ell+1)(\ell+3)}\right)$
The term $P_{\ell}(1)+P_{\ell}(-1)$ is zero for odd $\ell$ and $2 P_{\ell}(1)$ for even $\ell$. So we can rewrite our answer.
b. Calculate the surface charge density induced on the shell.

$$
\begin{gathered}
\sigma=-\epsilon_{0} \nabla \Phi \cdot \hat{n} \\
\sigma=-\frac{3 Q}{8 \pi} \sum_{\ell=0}^{\infty} P_{\ell}(\cos \theta) \frac{(2 \ell+1)}{(\ell+1)(\ell+3)} \frac{2}{b^{2}}\left(\frac{d}{b}\right)^{\ell}
\end{gathered}
$$

c. Discuss your answers to parts a and b in the limit $d \ll b$.

In this limit, the term $\left(\frac{d}{b}\right)^{\ell}$ except when $\ell=0$. Then,

$$
\sigma=-\frac{3 Q}{8 \pi} P_{0}(\cos \theta) \frac{1}{3} \frac{2}{b^{2}}=-\frac{Q}{4 \pi b^{2}}
$$

This is what we would have expected if a point charge were located at the origin and the sphere were at zero potential. When $d \ll b, r$ will most likely be greater than $d$ for the region of interest so it will suffice to take the limit of the second form for $\Phi$. Once again, only $\ell=0$ terms will contribute.

$$
\Phi=\frac{Q}{4 \pi \epsilon_{0}}\left(\frac{b-r}{b r}\right)
$$

This looks like the equation for a spherical capacitor's potential as it should!

## Problem 4.6

A nucleus with quadrupole moment $Q$ finds itself in a cylindrically symmetric electric field with a gradient $\left(\frac{\partial E_{z}}{\partial z}\right)_{0}$ along the $z$ axis at the position of the nucleus.
a. Find the energy of the quadrupole interaction.

Recall that the quadrapole tensor is

$$
Q_{i j}=\int\left(3 x_{i}^{\prime} x_{j}^{\prime}-r^{\prime 2} \delta_{i j}\right) \rho\left(\vec{r}^{\prime}\right) d V^{\prime}
$$

For the external field, Gauss's law tells us that a vanishing charge density means $\nabla \cdot \vec{E}=0$. $\frac{\partial E_{z}}{\partial z}+\frac{\partial E_{x}}{\partial x}+\frac{\partial E_{y}}{\partial y}=0$. The problem is cylindrically symmetric so $\frac{\partial E_{x}}{\partial x}=\frac{\partial E_{y}}{\partial y}=-\frac{1}{2} \frac{\partial E_{z}}{\partial z}$.
According to Jackson's equation 4.23, the energy for a quadrapole is

$$
W=-\frac{1}{6} \sum_{i=1}^{3} \sum_{j=1}^{3} \int\left(3 x_{i} x_{j}-r^{2} \delta_{i j}\right) \rho \frac{\partial E_{j}}{\partial x_{i}} d^{3} x
$$

When $i \neq j$, there is no contribution to the energy. You can understand this by recalling that the curl of $\vec{E}$ is zero for static configurations, i.e. $\nabla \times \vec{E}=0$. When $x_{i}=z$ and $x_{j}=z$, the integral is clearly $q Q_{33}=q Q_{\text {nucleus }}$, and the energy contribution is $W_{3}=-\frac{q}{6} Q \frac{\partial E_{z}}{\partial z}$. Jackson hints on page 151 that in nuclear physics $Q_{11}=Q_{22}=-\frac{1}{2} Q_{33}$. For $x_{i}$ or $x_{j}$ equals $x$ or $y, W=$ $\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{q}{6} Q \frac{\partial E_{z}}{\partial z}\right)=-\frac{q}{24} Q \frac{\partial E_{z}}{\partial z}$. Thus,

$$
W=-\left(\frac{1}{6}+\frac{1}{12}\right) q Q\left(\frac{\partial E_{z}}{\partial z}\right)=-\frac{q}{4} Q\left(\frac{\partial E_{z}}{\partial z}\right)
$$

q.e.d.
b. Calculate $\left(\frac{\partial E_{z}}{\partial z}\right)_{0}$ in units of $\frac{q}{4 \pi \epsilon_{0} a_{0}^{3}}$.

We are given $Q=2 \times 10^{-28} \mathrm{~m}^{2}, \mathrm{~W} / \mathrm{h}=10 \mathrm{MHz}, a_{0}=\frac{4 \pi \epsilon_{\epsilon} \hbar^{2}}{m_{e} q^{2}}=0.529 \times 10^{-10}$ $\mathrm{m}, \frac{q}{4 \pi \epsilon_{0} a_{0}^{3}}=9.73 \times 10^{2} \mathrm{~N} /(\mathrm{mC})$, and from part a,

$$
W=-\frac{q}{4} Q\left(\frac{\partial E_{z}}{\partial z}\right)
$$

Solve for $\left(\frac{\partial E_{z}}{\partial z}\right)$,

$$
\left(\frac{\partial E_{z}}{\partial z}\right)=\frac{W}{h}\left(\frac{-h}{q}\right) \frac{4}{1} \frac{1}{Q}
$$

Plugging in numbers, $8.27 \times 10^{20} \mathrm{~N} /(\mathrm{mC})$. In units of $\frac{q}{4 \pi \epsilon_{0} a_{0}^{3}}$ :

$$
\left(\frac{\partial E_{z}}{\partial z}\right)=8.5 \times 10^{-2}\left(\frac{q}{4 \pi \epsilon_{0} a_{0}^{3}}\right) \frac{N}{m \cdot C}
$$

c. Nuclear charge distributions can be approximated by a constant charge density throughout a spheroidal volume of semi-major axis $a$ and semi-minor axis $b$. Calculate the quadrupole moment of such a nucleus, assuming that the total charge is $Z q$. Given that Eu ${ }^{1} 53$ ( $Z=63$ ) has a quadrupole moment $Q=2.5 \times 10^{-28} \mathrm{~m}^{2}$ and a mean radius, $R=\frac{a+b}{2}=7 \times 10^{-15} \mathrm{~m}$, determine the fractional difference in the radius $\frac{a-b}{R}$.
For the nucleus, the total charge is $Z q$ where $q$ is the charge of the electron. The charge density is the total charge divided by the volume for points inside the nucleus. Outside the nucleus the charge density vanishes. The volume of an general ellipsoid is given by the high school geometry formula, $V=\frac{4}{3} \pi a b c$. In our case, $a$ is the semi-major axis, and $b$ and $c$ are the semi-minor axes. By cylindrical symmetry $b=c$.

$$
\rho=\left\{\begin{array}{l}
\frac{3 Z q}{4 \pi a b^{2}}, r \leq \frac{b}{a} \sqrt{a^{2}-z^{2}} \\
0, r>\frac{b}{a} \sqrt{a^{2}-z^{2}}
\end{array}\right.
$$

The nuclear quadrapole moment is defined $Q=\frac{1}{q} \int\left(3 z^{2}-R^{2}\right) \rho d V$. Because of the obvious symmetry, we'll do this in cylindrical coordinates where $R^{2}=$ $z^{2}+r^{2}$ and $d V=r d \theta d r d z$.

$$
Q=\frac{3 Z}{4 \pi a b^{2}} \int_{-a}^{a} \int_{0}^{\frac{b}{a} \sqrt{a^{2}-z^{2}}} \int_{0}^{2 \pi}\left(3 z^{2}-z^{2}-r^{2}\right) r d \theta d r d z
$$

The limits on the second integral are determined because the charge density vanishes outside the limits.

$$
Q=\frac{3 Z}{4 \pi a b^{2}} \int_{-a}^{a} \int_{0}^{\frac{b}{a} \sqrt{a^{2}-z^{2}}}\left(2 z^{2}-r^{2}\right) r d r d z
$$

Substitute $r^{2}=u$, and integrate over $d u$.

$$
\begin{array}{r}
Q=\frac{3 Z}{4 \pi a b^{2}} \int_{-a}^{a} \int_{0}^{b^{2}-\frac{z^{2} b^{2}}{a^{2}}}\left(2 z^{2}-u\right) d u d z \\
=\frac{3 Z}{4 \pi a b^{2}} \int_{-a}^{a}\left[2 z^{2}\left(b^{2}-\frac{z^{2} b^{2}}{a^{2}}\right)-\frac{1}{2}\left(b^{2}-\frac{z^{2} b^{2}}{a^{2}}\right)^{2}\right] d z
\end{array}
$$

Simplify.

$$
Q=\frac{3 Z}{4 a b^{2}} \int_{-a}^{a}\left[2 z^{2} b^{2}-\frac{2 z^{4} b^{2}}{a^{2}}-\frac{1}{2} b^{4}+\frac{b^{4} z^{2}}{a^{2}}-\frac{1}{2} b^{4} z^{4} a^{4}\right] d z
$$

Evaluate the next integral.

$$
Q=\frac{3 Z}{4 a b^{2}}\left[\frac{4 a^{3} b^{2}}{3}-\frac{4 b^{2} a^{3}}{5}-b^{4} a+\frac{2 b^{4} a}{3}-\frac{1}{5} a b^{4}\right]
$$

Simplify and factor.

$$
Q=\frac{2}{5} Z a^{2}-\frac{2}{5} z b^{2}=\frac{2 Z}{5}(a+b)(a-b)=\frac{8 Z}{5}\left(\frac{a+b}{2}\right)\left(\frac{a-b}{2}\right)
$$

Plug in $R=\frac{a+b}{2}$. $R$ is the mean radius, $7 \times 10^{-15}$ meters.

$$
Q=\frac{8 Z}{5} R\left(\frac{a-b}{2}\right)
$$

So finally, I can get what Jackson desires.

$$
\left(\frac{a-b}{2}\right)=\frac{5 Q}{8 Z R} \rightarrow \frac{a-b}{R}=\frac{5 Q}{8 Z} \frac{2}{R^{2}}
$$



## Problem 4.8

a very long, right circular, cylindrical shell of dielectric constant $\frac{\epsilon}{\epsilon_{0}}$ and inner and outer radii $a$ and $b$ respectively, is placed in a previously uniform electric field $\vec{E}_{0}$ with the cylinder's axis perpendicular to the field. The medium inside and outside the cylinder has a dielectric constant of unity. Determine the potential and electric field in the three regions, neglecting end effects.
Since the total charge is zero, we can use Poisson's equation:

$$
\nabla^{2} \Phi=0
$$

Symmetry in this problem leads me to choose cylindrical coordinates in which the Poisson equation is

$$
\frac{\partial^{2} \Phi}{\partial r^{2}}+\frac{1}{r} \frac{\partial \Phi}{\partial r}+\frac{1}{r^{2}} \frac{\partial^{2} \Phi}{\partial \theta^{2}}+\frac{\partial^{2} \Phi}{\partial z^{2}}=0
$$

Because of translational symmetry along the zaxis, $\Phi$ is independent of $z$, and we need only consider the problem in the $r-\theta$ plane.

$$
\frac{\partial^{2} \Phi}{\partial r^{2}}+\frac{1}{r} \frac{\partial \Phi}{\partial r}+\frac{1}{r^{2}} \frac{\partial^{2} \Phi}{\partial \theta^{2}}=0
$$

Try a separation of variables, i.e. $\Phi(r, \theta)=R(r) \Theta(\theta)$.

$$
\frac{r^{2}}{R}\left(\frac{\partial^{2} R}{\partial r^{2}}+\frac{1}{r} \frac{\partial R}{\partial r}\right)+\frac{1}{\Theta^{2}} \frac{\partial^{2} \Theta}{\partial \theta^{2}}=0
$$

This will give us two equations. The first isn't too hard to solve.

$$
\frac{\partial^{2} \Theta}{\partial \theta^{2}}=-m^{2} \Theta
$$

This admits the obvious solution:

$$
\Theta(\theta)=e^{ \pm i m \theta}
$$

We'll employ the convenient linear superposition

$$
\Theta(\theta)=\cos (m \theta)
$$

The second equation is a bit trickier.

$$
\frac{\partial^{2} R}{\partial r^{2}}+\frac{1}{r} \frac{\partial R}{\partial r}-\frac{m^{2}}{R^{2}} R=0
$$

Let's guess that

$$
R(r)=r^{ \pm m}, m \neq 0
$$

and

$$
R(r)=\ln r+C, m=0
$$

are solutions. In fact, it is not that hard to show that these are in fact solutions.
The general solution is a linear combination of these solutions. The boundary conditions will determine just what this linear combination is. The uniform external field can be reproduced by $\Phi=-E_{0} r \cos \phi$. At the surface of the cylinder we have another boundary condition. Namely, at $x=a$ or $b, \vec{E}_{\|}$ and $\vec{D}_{\perp}$ are continuous. Recall that $E_{\|}=-\frac{\partial \Phi}{\partial \theta}$ and $\epsilon E_{\perp}=-\epsilon \frac{\partial \Phi}{\partial r}$. On physical grounds, we can limit the form of the solution outside and inside the cylindrical region. Outside, we need to have the electric field at infinity, but we certainly don't want the field to diverge. The logarithmic and $r^{n}$ with $n>1$ terms diverge as $r$ goes to infinity; clearly, these terms are unphysical.

$$
\Phi_{\text {out }}=-E_{0} r \cos \theta+\sum_{m=1}^{\infty} A_{m} \frac{1}{r^{m}} \cos \left(m \theta+\alpha_{m}\right)
$$

In between the cylindrical shells, we don't have any obvious physical constraints.

$$
\Phi_{m i d}=\sum_{m=1}^{\infty} B_{-m} \frac{1}{r^{m}} \cos \left(m \theta+\beta_{-m}\right)+\sum_{m=1}^{\infty} B_{m} r^{m} \cos \left(m \theta+\beta_{m}\right)+C \ln r
$$

Inside, we have to eliminate the diverging terms at the origin.

$$
\Phi_{i n}=\sum_{m=1}^{\infty} D_{m} r^{m} \cos \left(m \theta+\delta_{m}\right)
$$

Now, it's time to match boundary conditions. They were $\Phi=-E_{0} r \cos \phi$ as $x \rightarrow \infty$, and at $x=a$ or $b ; E_{\|}=-\frac{\partial \Phi}{\partial \theta}$ and $\epsilon E_{\perp}=-\epsilon \frac{\partial \Phi}{\partial r}$ are continuous. For $m \neq 1$, we find $\alpha_{m}=\beta_{m}=\beta_{-m}=\delta_{m}$ and $A_{m}=B_{m}=B_{-m}=D_{m}=0$. We might have suspected that only the $m=0$ terms contribute because the only thing that breaks the symmetry in this problem is the external electric field which has $m=1$. Note further that for $m=0, A_{0}=0$, and $C=0$. I am left with the following forms:
Outside:

$$
\Phi_{\text {out }}=-E_{0} r \cos \theta+A_{1} \frac{1}{r} \cos \left(\theta+\alpha_{1}\right)
$$

In between the cylinders:

$$
\Phi_{\text {mid }}=B_{-1} \frac{1}{r} \cos \left(\theta+\beta_{-1}\right)+B_{1} r \cos \left(\theta+\beta_{1}\right)
$$

Inside the cylinders:

$$
\Phi_{i n}=D_{1} r \cos \left(\theta+\delta_{1}\right)
$$

Because each region has the same symmetry with respect to the external field, we can drop the phases. For the outside region, we find

$$
\Phi_{\text {out }}=\left(-E_{0} r+A_{1} \frac{1}{r}\right) \cos (\theta)
$$

And likewise in between, we have

$$
\Phi_{\text {mid }}=\left(B_{-1} \frac{1}{r}+B_{1} r\right) \cos (\theta)
$$

And inside,

$$
\Phi_{i n}=D_{1} r \cos (\theta)
$$

Applying the other boundary condition, $\frac{\partial \Phi}{\partial \theta}$, we get Outside:

$$
-E_{0} b+A_{1} \frac{1}{b}=B_{1} b+B_{-1} \frac{1}{b}
$$

And

$$
B_{1} a+\frac{B_{-1}}{a}=D_{1} a
$$

For the final boundary condition, $\epsilon \frac{\partial \Phi}{\partial r}$ :

$$
\begin{gathered}
-\epsilon_{0}\left(E_{0}+\frac{A_{1}}{b^{2}}\right)=\epsilon\left(B_{1}-B_{-1} \frac{1}{b^{2}}\right) \\
\epsilon\left(B_{-1}-B_{1} \frac{1}{a^{2}}\right)=\epsilon_{0} D_{1}
\end{gathered}
$$

Let $\frac{\epsilon}{\epsilon_{0}}=\kappa$, the capacitance. Solve simultaneously.

$$
\begin{gathered}
A_{1}=E_{0} b^{2}+2 E_{0} b^{2} \frac{a^{2}(1-\kappa)-b^{2}(1+\kappa)}{b^{2}(1+\kappa)^{2}-a^{2}(1-\kappa)^{2}} \\
B_{1}=\frac{-2 E_{0} b^{2}(1+\kappa)}{b^{2}(1+\kappa)^{2}-a^{2}(1-\kappa)^{2}} \\
B_{-1}=\frac{2 E_{0} a^{2} b^{2}(1-\kappa)}{b^{2}(1+\kappa)^{2}-a^{2}(1-\kappa)^{2}} \\
D_{1}=\frac{-4 E_{0} b^{2}}{b^{2}(1+\kappa)^{2}-a^{2}(1-\kappa)^{2}}
\end{gathered}
$$

b. Sketch the lines of force for a typical case of $b \simeq 2 a$.

Since we are only concerned with a qualitative sketch, we'll consider a particular case. Take $\kappa=3, E_{0}=2$ and $a^{2}=2$. Then, we have $A_{1}=-2$, $B_{1}=-2, B_{-1}=-2$, and $D_{1}=-1$. The potential becomes

$$
\begin{aligned}
& \Phi_{\text {out }}=-E r \cos \theta-\frac{2}{r} \cos \theta \\
& \Phi_{\text {mid }}=-2 r \cos \theta-\frac{2}{r} \cos \theta
\end{aligned}
$$

And

$$
\Phi_{i n}=-r \cos \theta
$$

Now, I just need to make the plots.
c. Discuss the limiting forms of your solution appropriate for a solid dielectric cylinder in a uniform field, and a cylindrical cavity in a uniform dielectric.

For the dielectric cylinder, I shrink the inner radius down to nothing; $a \rightarrow 0$.

$$
\begin{gathered}
A_{1}=\frac{\kappa-1}{\kappa+1} b^{2} E_{0} \\
B_{1}=\frac{-2 E_{0}}{1+\kappa} \\
B_{-1}=0 \\
D_{1}=\frac{-4 E_{0}}{(1+\kappa)^{2}}
\end{gathered}
$$

For the cylindrical cavity, I place the surface of the outer shell at infinity, $b \rightarrow \infty$. In this limit $A_{1}$ is ill-defined, so we'll ignore it.

$$
\begin{gathered}
B_{1}=\frac{-2 E_{0}}{1+\kappa} \\
B_{-1}=2 E_{0} a^{2} \frac{1-\kappa}{(1+\kappa)^{2}} \\
D_{1}=\frac{-4 E_{0}}{(1+\kappa)^{2}}
\end{gathered}
$$



## Problem 4.10

Two concentric conducting spheres of inner and outer radii $a$ and $b$, respectively, carry charges $\pm Q$. The empty space between the spheres is half-filled by a hemispherical shell of dielectric (of dielectric constant $\kappa=\frac{\epsilon}{\epsilon_{0}}$ ), as shown in the figure.
a. Find the electric field everywhere between the spheres.

We use what I like to call the $D$ law, that is $\nabla \cdot \vec{D}=\rho_{\text {free }}$. The divergence theorem tells us

$$
\oint \vec{D} \cdot d \vec{A}=Q
$$

Because of this radial symmetry, we expect that $E_{\theta}$ and $E_{\phi}$ will vanish, and by Gauss's law, we expect $E_{r}$ to be radially symmetric. Therefore, we need only to find the radial components of $\vec{D}$ recalling that $\vec{E}=\epsilon \vec{D}$. Use the $D$ theorem and that $\vec{D}=\epsilon \vec{E}$.

$$
\epsilon_{0} E_{r} 2 \pi r^{2}+\epsilon E_{r} 2 \pi r^{2}=Q
$$

This gives an electric field:

$$
\vec{E}=\left(\frac{2}{1+\frac{\epsilon}{\epsilon_{0}}}\right) \frac{Q}{4 \pi \epsilon_{0} r^{2}} \hat{r}
$$

This has the form of Coulomb's law but with an effective total charge, $Q_{e f f}=$ $\frac{2 \epsilon_{0}}{\epsilon+\epsilon_{0}} Q$.
b. Calculate the surface charge distribution on the inner sphere. $\sigma_{i}=\epsilon_{i} E_{r}$ in this case. On the inner surface,

$$
\sigma_{\text {dielectric }}=\left(\frac{\epsilon}{\epsilon_{0}+\epsilon}\right) \frac{Q}{2 \pi a^{2}}
$$

And

$$
\sigma_{a i r}=\left(\frac{\epsilon_{0}}{\epsilon_{0}+\epsilon}\right) \frac{Q}{2 \pi a^{2}}
$$

c. Calculate the polarization charge density induced on the surface of the dielectric at $r=a$.
Find the polarization charge density by subtracting the effective charge density from the total contained charge density: $Q_{e f f}=Q+Q_{p o l}$. This gives $Q_{\text {polarization }}=\left(\frac{\epsilon_{0}-\epsilon}{\epsilon_{0}+\epsilon}\right) Q$. The total charge density is obtained by averaging the polarization charge over the half the inner sphere's surface which is in contact with the dielectric. $\sigma_{\text {polarization }}=\frac{Q_{\text {polarization }}}{2 \pi a^{2}}$. Therefore, the polarization charge density is:

$$
\sigma_{\text {polarization }}=-\left(\frac{\epsilon_{0}-\epsilon}{\epsilon_{0}+\epsilon}\right) \frac{Q}{2 \pi a^{2}}
$$

An alternative way of finding this result is to consider the polarization, $\vec{P}=$ $\left(\epsilon-\epsilon_{0}\right) \vec{E}$. Jackson argues that $\sigma_{\text {polarization }}=\vec{P} \cdot \vec{n}$. But $\vec{P}$ points from the dielectric outward at $r=a$, and $\sigma_{\text {polarization }}=-P_{r}=\left(\epsilon_{0}-\epsilon\right) E_{r}=\left(\frac{\epsilon_{0}-\epsilon}{\epsilon_{0}+\epsilon}\right) \frac{Q}{2 \pi a^{2}}$ as before.

## Problem 5.1

Starting with the differential expression

$$
d \vec{B}=\frac{\mu_{0}}{4 \pi} I d \overrightarrow{\ell^{\prime}} \times\left(\frac{\vec{r}-\vec{r}^{\prime}}{\left|\vec{r}-\vec{r}^{\prime}\right|^{3}}\right)
$$

for the magnetic induction at a point $P$ with the coordinate $\vec{x}$ produced by an increment of current $I d \ell^{\prime}$ at $\vec{x}^{\prime}$, show explicitly that for a closed loop carrying a current $I$ the magnetic induction at $P$ is

$$
\vec{B}=\frac{\mu_{0}}{4 \pi} I \vec{\nabla} \Omega
$$

where $\Omega$ is the solid angle subtended by the loop at the point $P$. This corresponds to a magnetic scalar potential, $\Phi_{M}=\frac{-\mu_{0} I \Omega}{4 \pi}$. The sign convention for the solid angle is that $\Omega$ is positive if the point $P$ views the "inner" side of the surface spanning the loop, that is, if a unit normal $\vec{n}$ to the surface is defined by the direction of current flow via the right hand rule, $\Omega$ is positive if $\vec{n}$ points away from the point $P$, and negative otherwise.
Biot-Savart's law tells us how to find the magnetic field at some point $P(\vec{r})$ produced by a wire element at some other point $P_{2}\left(\overrightarrow{r^{\prime}}\right)$. At $P(\vec{r})$ :

$$
d \vec{B}=\frac{\mu_{0}}{4 \pi} I d \overrightarrow{\ell^{\prime}} \times\left(\frac{\vec{r}-\vec{r}^{\prime}}{\left|\vec{r}-\vec{r}^{\prime}\right|^{3}}\right)
$$

The total $\vec{B}$-field at a point $P$ is the sum of the $d \vec{B}$ elements from the entire loop. So we integral $d \vec{B}$ around the closed wire loop.

$$
\vec{B}=\int d \vec{B}=\frac{\mu_{0}}{4 \pi} I \oint_{\Gamma} d \ell^{\prime} \times\left(\frac{\vec{r}-\vec{r}^{\prime}}{\left|\vec{r}-\vec{r}^{\prime}\right|^{3}}\right)
$$

There is a form of Stokes' theorem which is useful here: $\oint d \ell^{\prime} \times A=\int d S^{\prime} \times$ $\nabla^{\prime} \times A$. I'll look up a definitive reference for this someday; this maybe on the inside cover of Jackson's book.

$$
\oint_{\Gamma} d \ell^{\prime} \times\left(\frac{\vec{r}-\vec{r}^{\prime}}{\left|\vec{r}-\vec{r}^{\prime}\right|^{3}}\right)=\int d S^{\prime} \times \nabla^{\prime} \times\left(\frac{\vec{r}-\vec{r}^{\prime}}{\left|\vec{r}-\vec{r}^{\prime}\right|^{3}}\right)
$$

With the useful identity, $\nabla^{\prime} f\left(x-x^{\prime}\right)=\nabla f\left(x^{\prime}-x\right)$, we have

$$
d S^{\prime} \times \nabla^{\prime} \times\left(\frac{\vec{r}-\vec{r}^{\prime}}{\left|\vec{r}-\vec{r}^{3}\right|^{3}}\right)=d S^{\prime} \times \nabla \times\left(\frac{\vec{r}^{\prime}-\vec{r}}{\left|\vec{r}^{\prime}-\vec{r}\right|^{3}}\right)
$$

Now, with the use the vector identity, $\vec{A} \times(\vec{B} \times \vec{C})=(A \cdot C) \vec{B}-(A \cdot B) \vec{C}$, I can write the triple cross product under the integral as two terms. The integral becomes

$$
\vec{B}=-\frac{\mu_{0}}{4 \pi} I \int \nabla \cdot\left(\frac{\vec{r}^{\prime}-\vec{r}}{|\vec{r}-\vec{r}|^{3}}\right) d \vec{S}^{\prime}+\frac{\mu_{0}}{4 \pi} I \int \vec{\nabla}\left[\left(\frac{\vec{r}^{\prime}-\vec{r}}{\left|\vec{r}^{\prime}-\vec{r}\right|^{3}}\right) \cdot d S^{\prime}\right]
$$

But $\left(\frac{\overrightarrow{r^{\prime}}-\vec{r}}{\mid \vec{r}^{\prime}-\vec{r}^{3}}\right)=\nabla\left(\frac{1}{\left|\vec{r}-\vec{r}^{\prime}\right|}\right)$ and $\nabla^{2}\left(\frac{1}{\left|\vec{r}-\vec{r}^{\prime}\right|}\right) \propto \delta\left(r^{\prime}-r\right)$. The first integral vanishes on the surface where $r^{\prime}$ does not equal $r$. Since I am free to choose any area which is delimited by the closed curve $\Gamma$, I choose a surface so that $r^{\prime}$ does not equal $r$ on the surface, and the first term vanishes. We are left with

$$
\vec{B}=\frac{\mu_{0}}{4 \pi} I \vec{\nabla} \int\left[\left(\frac{\vec{r}^{\prime}-\vec{r}}{\left|\vec{r}^{\prime}-\vec{r}\right|^{3}}\right) \cdot d S^{\prime}\right]
$$

I take $\nabla$ outside of the integral because the integral does not depend on $r^{\prime}$, as the integration does.
An element of solid angle is an element of the surface area, $a \vec{A} \cdot \hat{R}$, of a sphere divided by the square of that sphere's radius, $R^{2}$, so that the solid angle has dimension-less units (so called steradians). To get a solid angle, we integral over the required area.

$$
\Omega=\int_{A} \frac{d \vec{A} \cdot(\hat{R})}{R^{2}}=\int \frac{\vec{R} \cdot d \vec{A}}{R^{3}}
$$

And in our notation, this is

$$
\Omega=\int\left[\left(\frac{\vec{r}^{\prime}-\vec{r}}{\left|\overrightarrow{r^{\prime}}-\vec{r}\right|^{3}}\right) \cdot d \overrightarrow{S^{\prime}}\right]
$$

Thus,

$$
\vec{B}=\frac{\mu_{0}}{4 \pi} I \vec{\nabla} \Omega
$$

Where $\Omega$ is the solid angle viewed from the observation point subtended by the closed current loop.

## Problem 5.9

A current distribution $\vec{J}(\vec{x})$ exists in a medium of unit relative permeability adjacent to a semi-indefinite slab of material having relative permeability $\mu_{r}$ and filling the half-space, $z<0$.
a. Show that for $z>0$, the magnetic induction can be calculated by replacing the medium of permeability $\mu_{r}$ by an image current distribution, $\vec{J}^{*}$, with components as will be derived.
Well, this problem is not too bad. Jackson solved this for a charge distribution located in a dielectric $\epsilon_{1}$ above a semi-infinite dielectric plane with $\epsilon_{2}$.

$$
\begin{gathered}
q^{*}=-\left(\frac{\epsilon_{2}-\epsilon_{1}}{\epsilon_{2}+\epsilon_{1}}\right) q \\
q^{* *}=\left(\frac{2 \epsilon_{2}}{\epsilon_{2}+\epsilon_{1}}\right) q
\end{gathered}
$$

With some careful replacements, we can generalize these to solve for the image currents. We will consider point currents, whatever the Hell they are. Physically, a point current makes no sense and violates the conservation of charge, but mathematically, it's useful to pretend such a thing could exist. Associate each component of $\vec{J}(x, y, z)$ with $q$. Set $\epsilon_{1} \rightarrow \mu_{1}=1$ and $\epsilon_{2} \rightarrow$ $\mu_{2}=\mu$. These replacements give us the images modulo a minus sign.
For $z>0$, we have to be careful about the overall signs of the image currents. We can find the signs by considering the limiting cases of diamagnetism and paramagnetism. That is when $\mu \rightarrow 0$, we have paramagnetism, and when $\mu \rightarrow \infty$, we have diamagnetism. Let's work with the diamagnetic case. The image current will reduce the effect of the real current. Using the right hand rule, we'd expect parallel wires to carry the current in the same direction for this case. Therefore, we must have

$$
\vec{J}_{\|}^{*}=\left(\frac{\mu-1}{\mu+1}\right) \vec{J}_{\|}
$$

The perpendicular part of the image current, on the other hand, must flow in the opposite direction of the real current.

$$
J_{\perp}^{*}=-\left(\frac{\mu-1}{\mu+1}\right) J_{\perp}
$$

We can understand this using an argument about mirrors. For the parallel components, the image currents must be parallel and in the same direction
for the diamagnetic case. Think of a mirror and the image of your right hand in a mirror. If you move your right hand to the right, its image also moved to the right. If it's a dirty mirror, then a dim image of our hand moves to the right. That's exactly what we'd expect. The same direction current, but smaller magnitude. For the perpendicular point current, we need an opposite sign (anti-parallel) image current. This is not much more difficult to visualize. Think of the image of your right hand in mirror. Move your hand away from you, and watch its image move towards you!
If we had considered the paramagnetic case, the image currents would reverse direction. This is because we now want the images to contribute to the fields caused by the real currents. The sign flip changes two competing currents to two collaborating currents.
b. Show that for $z<0$ the magnetic induction appears to be due to a current distribution $\left[2 \mu_{r} /\left(\mu_{r}+1\right)\right] \vec{J}$ is a medium of unit relative permeability.
For $z<0$ : Once again, we associate $\vec{J}$ with $q$ to find $\vec{J}^{* *}$. Set $\mu_{1}=1$ and $\mu_{2}=\mu$. Notice that all the signs are positive. For the components of the current parallel to the surface, this is exactly as expected. For the $z$ component, we have a reflection of a reflection or simply a weakened version of the original current as our image; therefore, the sign is positive.

$$
\vec{J}^{* *}=\left(\frac{2 \mu}{\mu+1}\right) \vec{J}
$$

To get a better understanding of the physics involved here, I will derive these results using the boundary conditions. We are solving

$$
\nabla \times \vec{H}=\vec{J}
$$

Which has a formal integral solution

$$
\vec{H}=\frac{\mu_{0}}{4 \pi \mu} \int d^{3} \vec{r}^{\prime} \vec{J}\left(r^{\prime}\right) \times \frac{\left|\vec{r}-\overrightarrow{r^{\prime}}\right|}{\left|r-r^{\prime}\right|^{3}}
$$

But our $J$ s are point currents, that is $\vec{J} \propto \delta(\vec{r}-\vec{a})$, so we can do the integral and write

$$
\vec{H} \propto \frac{1}{\mu} \vec{I} \times\left(\vec{r}-\vec{a}^{\prime}\right)
$$

The cross product is what causes all the trouble. We will choose $\vec{r}= \pm \hat{k}$, $\vec{a}^{\prime}=\hat{j}, \vec{I}=I_{x} \hat{i}+I_{y} \hat{j}+I_{z} \hat{k}, \overrightarrow{I^{*}}=I_{x}^{*} \hat{i}+I_{y}^{*} \hat{j}+I_{z}^{*} \hat{k}$, and $\vec{I}^{* *}=I_{x}^{* *} \hat{i}+I_{y}^{* *} \hat{j}+I_{z}^{* *} \hat{k}$.
Notice that I have not made any assumptions about the signs of the various image current components. Then,

$$
\begin{array}{r}
\vec{H} \propto-\left(I_{y}+I_{z}\right) \hat{i}+I_{x} \hat{j}+I_{x} \hat{k} \\
\vec{H}^{*} \propto\left(I_{y}^{*}-I_{z}^{*}\right) \hat{i}-I_{x}^{*} \hat{j}+I_{x}^{*} \hat{k}
\end{array}
$$

And

$$
\vec{H}^{* *} \propto \frac{1}{\mu}\left(I_{y}^{* *}-I_{z}^{* *}\right) \hat{i}-\frac{1}{\mu} I_{x}^{* *} \hat{j}+\frac{1}{\mu} I_{x}^{* *} \hat{k}
$$

We have the boundary conditions: 1 .

$$
\vec{B}_{2} \cdot \hat{n}=\vec{B}_{1} \cdot \hat{n} \rightarrow \mu_{2} \vec{H}_{2} \cdot \hat{n}=\mu_{1} \vec{H}_{1} \cdot \hat{n}
$$

And 2.

$$
\vec{H}_{2} \times \hat{n}=\vec{H}_{1} \times \hat{n}
$$

Note $\hat{n}=\hat{k}$. From the first condition:

$$
I_{x}+I_{x}^{*}=I_{x}^{* *}
$$

From the other condition, we find for the $\hat{i}$ component

$$
-\frac{1}{\mu} I_{x}^{* *}=-\left(I_{x}-I_{x}^{*}\right)
$$

Solve simultaneously to find

$$
I_{x}^{* *}=\frac{2 \mu}{\mu+1} I_{x}
$$

And

$$
I_{x}^{*}=\frac{\mu-1}{\mu+1} I_{x}
$$

By symmetry, we know that these equations still hold with the replacement $x \rightarrow y$. We have one more condition left from the $\hat{j}$ component.

$$
-\frac{1}{\mu}\left(I_{y}^{* *}-I_{z}^{* *}\right)=\left(I_{z}+I_{z}^{*}\right)-\left(I_{y}-I_{y}^{*}\right)
$$

To make life easier, we'll put $I_{y}$ to zero. Then,

$$
\frac{1}{\mu} I_{z}^{* *}=I_{z}+I_{z}^{*}
$$

I'm not sure how to get a unique solution out of this, but if I assume that $I_{z}^{* *}$ has the same form as $I_{x}^{* *}, I$ find

$$
I_{z}^{*}=\frac{1-\mu}{\mu+1} I
$$

## Problem 6.11

A transverse plan wave is incident normally in a vacuum on a perfectly absorbing flat screen.
a. From the law of conservation of linear momentum, show that the pressure (called radiation pressure) exerted on the flat screen is equal to the field energy per unit volume in the wave.
The momentum density for a plane wave is $\overrightarrow{\mathcal{P}}=\frac{1}{c^{2}} \vec{S}$ with the Poynting vector, $\vec{S}=\frac{1}{\mu_{0}} \vec{E} \times \vec{B}$. The total momentum is the momentum density integrated over the volume in question.

$$
\vec{p}=\int \overrightarrow{\mathcal{P}} d V \sim \overrightarrow{\mathcal{P}} A d x \hat{x}
$$

The last step is true assuming $\overrightarrow{\mathcal{P}}$ does not vary much over the volume in question. Be aware that $d V=A d x$, the volume element in question. By Newton's second law, the force exerted in one direction (say $x$ ) is

$$
F_{x}=\frac{d p_{x}}{d t}=\frac{d}{d t}(\mathcal{P} A d x)=\mathcal{P} A \frac{d x}{d t}=\mathcal{P} A c
$$

$c$ is the speed of light. After all, electro-magnetic waves are just light waves. We want pressure which is force per unit area.

$$
\vec{P}=\frac{\vec{F}}{A}=\overrightarrow{\mathcal{P}} c=\frac{1}{c} \vec{S}=\frac{1}{c \mu_{0}} \vec{E} \times \vec{B}
$$

Take the average over time, and factor of one half comes in. We also know that $B_{0}=\frac{E_{0}}{c}$. Then,

$$
P=\frac{1}{2} \frac{1}{\mu_{0} c^{2}}\left(E_{0}\right)^{2}
$$

But $c^{2}=\frac{1}{\mu_{0} \epsilon_{0}}$, so

$$
P=\frac{1}{2} \epsilon_{0}\left(E_{0}\right)^{2}
$$

We already know from high school physics or Jackson equation 6.106 that the energy density is $\frac{1}{2}(\vec{E} \cdot \vec{D}+\vec{B} \cdot \vec{H}) \rightarrow \frac{1}{2} \epsilon_{0}\left(E_{0}\right)^{2}$ and wait that's the same as the pressure!

$$
P=\frac{1}{2} \epsilon_{0}\left(E_{0}\right)^{2}=u
$$

This result generalizes quite easily to the case of a non-monochromatic wave by the superposition principle and Fourier's theorem.
b. In the neighborhood of the earth the flux of electro-magnetic energy from the sun i approximately $1.4 \mathrm{~kW} / \mathrm{m}^{2}$. If an interplanetary "sail-plane" had a sail of mass $1 \mathrm{~g} / \mathrm{m}^{2}$ per area and negligible other weight, what would be its maximum acceleration in meters per second squared due to the solar radiation pressure? how does this compare with the acceleration due to the solar "wind" (corpuscular radiation)?
Energy Flux from the Sun: $1.4 \mathrm{~kW} / \mathrm{m}^{2}$
Mass/Area of Sail: $0.001 \mathrm{~kg} / \mathrm{m}^{2}$
The force on the sail is the radiation pressure times the sail area. In part a, we discovered that the electro-magnetic radiation pressure is the same as the energy density. Thus, $F=P A=u A$. Now, by Newton's law $F=m a$. The energy density is $\Phi / c$ where $\Phi$ is the energy flux given off by the sun. The acceleration of the sail is $\frac{\Phi}{c} \frac{A}{m}=14000 \div\left(3 \times 10^{8}\right) \times 1000=4.6 \times 10^{-3}$ $\mathrm{m} / \mathrm{sec}^{2}$.
According to my main man, Hans C. Ohanian, the velocity of the solar wind is about $400 \mathrm{~km} / \mathrm{sec}$. I'll guess-timate the density of solar wind particles as one per cubic centimeter ( $\rho=$ particles/volume $*$ mass $/$ particle $=1.7 \times 10^{-21}$ $\mathrm{kg} / \mathrm{m}^{3}$ ). Look in an astro book for a better estimate.

$$
\Delta p=\mathcal{P} A v \Delta t
$$

Clearly, $\mathcal{P}=\rho v$. The change in the momentum of the sail is thus $a=\frac{1}{m} \frac{\Delta p}{\Delta t}=$ $\mathcal{P} A v / m=\rho \frac{A}{m} v^{2}$. Numerically, we find $a=2.7 \times 10^{-9} \mathrm{~m} / \mathrm{sec}^{2}$.
Evidently, we can crank more acceleration out of a radiation pressure space sail ship than from a solar wind powered one.

## Problem 6.15

If a conductor or semiconductor has current flowing in it because of an applied electric field, and a transverse magnetic field is applied, there develops a component of electric field in the direction orthogonal to both the applied electric field (direction of current flow) and the magnetic field, resulting in a voltage difference between the sides of the conductor. This phenomenon is known as the Hall effect.
a. Use the known properties of electro-magnetic fields under rotations and spatial reflections and the assumption of Taylor series expansions around zero magnetic field strength to show that for an isotropic medium the generalization of Ohm's law, correct to second order in the magnetic field, must have the form:

$$
\vec{E}=\rho_{0} \vec{J}+R(\vec{H} \times \vec{J})+a_{2 a} \vec{H}^{2} \vec{J}+a_{2 b}(\vec{H} \cdot \vec{J}) \vec{H}
$$

where $\rho_{0}$ is the resistivity in the absence of the magnetic field and $R$ is called the Hall coefficient.
In Jackson section 6.10, Jackson performs a similar expansion for $\vec{p}$. We'll proceed along the same lines.
The zeroth term is, $a_{0} \vec{J}$ (Ohm's law). This is the simplest combination of terms which can still give us a polar vector.
Because $\vec{E}$ is a polar vector, the vector terms on the right side on the equation must be polar. $\vec{H}$ is axial so it alone is not allowed, but certain cross products and dot products produce polar vectors and are allowed. They are
First Order: $a_{1}(\vec{H} \times \vec{J})$
Second Order: $a_{2 a}(\vec{H} \cdot \vec{H}) \vec{J}+a_{2 b}(\vec{H} \cdot \vec{J}) \vec{H}$
When $\vec{H}=0, \vec{E}=\rho_{0} \vec{J}$ so $a_{1}=\rho_{0}$. And

$$
\vec{E}=\rho_{0} \vec{J}+R(\vec{H} \times \vec{J})+a_{2 a} \vec{H}^{2} \vec{J}+a_{2 b}(\vec{H} \cdot \vec{J}) \vec{H}
$$

I let $a_{1}=R$.
b. What about the requirements of time reversal invariance?

Under time reversal, we have a little problem. $\vec{E}$ and $\rho_{0}$ are even but $\vec{J}$ is odd. But then again if you think about it things really aren't that bad. Ohm's law is a dissipative effect and we shouldn't expect it to be invariant under time reversal.

## Problem 6.16

a. Calculate the force in Newton's acting a Dirac mono-pole of the minimum magnetic charge located a distance $\frac{1}{2}$ an Angstrom from and in the median plane of a magnetic dipole with dipole moment equal to one nuclear magneton.
A magnetic dipole, $\vec{m}=\frac{q \hbar}{2 m_{p}}$, creates a magnetic field, $\vec{B}$.

$$
\vec{B}=\frac{\mu_{0}}{4 \pi}\left[\frac{3 \vec{n}(\vec{n} \cdot \vec{m})-\vec{m}}{|\vec{x}|^{3}}\right]
$$

Along the meridian plane, $\vec{n} \cdot \vec{m}=0$ so

$$
\vec{B}=-\frac{\mu_{0}}{4 \pi} \frac{\vec{m}}{|\vec{x}|^{3}}
$$

Suppose this field is acting on a magnetic mono-pole with charge $g=\frac{2 \pi \hbar}{q} n$. Where $n$ is some quantum number which we'll suppose to be 1 . The force is

$$
F=-\frac{\mu_{0}}{4 \pi} \frac{g|m|}{|x|^{3}} \hat{m}
$$

And the magnitude

$$
|F|=\frac{\mu_{0}}{4 \pi} \frac{g q \hbar}{2 m_{p} r^{3}}=\frac{\mu_{0} \hbar^{2}}{4 m_{p} r^{3}}=2 \times 10^{-11} \text { Newtons }
$$

where we use $r=0.5$ Angstroms.
b. Compare the force in part a with atomic forces such as the direct electrostatic force between charges (at the same separation), the spin-orbit force, the hyper-fine interaction. Comment on the question of binding of magnetic mono poles to nuclei with magnetic moments. Assume the mono-poles mass is at least that of a proton. The electro static force at the same separation if given by Coulumb's law. $|F|=\frac{1}{4 \pi \epsilon_{0}} \frac{q^{2}}{r}=9.2 \times 10^{-8}$ Newtons where I have used $\frac{1}{4 \pi \epsilon_{0}}=8.988 \times 10^{9}$ $\mathrm{Nm}^{2} / \mathrm{C}, q=1.602 \times 10^{-19} \mathrm{C}$, and $r=0.5$ Angstroms. The fine structure is approximately $\alpha=\frac{1}{137}$ times the Coulomb force, so we expect this contribution to be about $7 \times 10^{-10}$ Newtons. The hyperfine interaction is smaller by a factor of $\frac{m_{e}}{m_{p}}=\frac{1}{1836}$, so $F_{f s} \simeq 4 \times 10^{-13}$ Newtons. I guess we should be able to see the effects on magnetic mono-poles on nuclei if those mono-poles exist. Unless of course mono-poles or super-massive. Or perhaps mono-poles are endowed with divine attributes which make them terribly hard to detect.

## Problem 7.2

A plane wave is incident on a layered interface as shown in the figure. The indices of refraction of the three non-permeable media are $n_{1}, n_{2}$, and $n_{3}$. The thickness of the media layer is $d$. Each of the other media is semi-infinite. a. Calculate the transmission and re-

flection coefficients (ratios of transmitted and reflected Poynting's flux to the incident flux), and sketch their behavior as a function of frequency for $n_{1}=1, n_{2}=2, n_{3}=3 ; n_{1}=3, n_{2}=2, n_{3}=1$; and $n_{1}=2, n_{2}=4, n_{3}=1$.
Look at the diagram. We have three layers of material labeled $I, I I$, and $I I I$ respectively. Each layer has a corresponding index of refraction, $n_{1}, n_{2}$, and $n_{3}$. An electro-magnetic wave is incident from the left and travels through the layers in the sequence $I \rightarrow I I \rightarrow I I I$. Because this is an electro-magnetic wave, we know $\vec{k} \times \hat{N}=0$ and $\vec{B} \cdot \vec{E}=0$. That is the $E$ and $B$ fields are perpendicular to the motion of the wave and are mutually perpendicular. These are non-permeable media so $\mu_{1}=\mu_{2}=\mu_{3}=1$ and $n_{i}=n\left(\epsilon_{i}\right)$ only. To find the effective coefficient of reflection, we will consider closely what is going on. The first interface can reflect the wave and contribute directly to the effective reflection coefficient, or the interface can transmit the wave. The story's not over yet because the second interface can also reflect the wave. If the wave is reflected, it will travel back to the first interface where it could be transmitted back through the first interface. Or the wave could bounce back. The effective reflection coefficient will be an infinite series of terms. Each subsequent term corresponds to a certain number of bounces between surface A and surface B before the wave is finally reflected to the left.

$$
r=r_{12}+t_{12} r_{23} t_{21} e^{2 i k_{2} d}+t_{12} r_{23} r_{21} t_{21} e^{4 i k_{2} d}+\ldots
$$

The first term corresponds to the reflection at the $n_{1}-n_{2}$ interface. The second term is a wave which passes through the $n_{1}-n_{2}$ interface, reflects off the $n_{2}-n_{3}$ interface, and then transits through the $n_{1}-n_{2}$ interface. Higher order terms correspond to multiple internal reflections.
Note the phase change over the internal reflection path. It is $2 k_{2} d$ for one round trip from the $n_{1}-n_{2}$ to the $n_{2}-n_{3}$ interface and back. $k_{2}$ is $\frac{n_{2} \omega}{c}$. The phase shift makes sense because the term in the exponent is really $i \vec{k} \cdot \vec{r}$ and the distance is $\vec{r}=\vec{d}$ for the first leg. On the return leg, the sign of both $\vec{k}$ and $\vec{d}$ change because the wave is propagating backwards and over the same distance in the opposite direction as before. The total phase change is the product of these two changes and so $i k d+i(-k)(-d)=2 i k d$. If you are motivated, you could probably show this with matching boundary conditions. I think this heuristic argument suffices.
I'll write this series in a suggestive form:

$$
r=r_{12}+\left[t_{12} r_{23} t_{21} e^{2 i k_{2} d}\right] \times\left[1+r_{23} r_{21} e^{2 i k_{2} d}+\left(r_{23} r_{21}\right)^{2} e^{4 i k_{2} d}+\ldots\right]
$$

The second term in the brackets is a geometric series:

$$
\sum_{n=0}^{\infty} x^{n}=\frac{1}{1-x}, x<1
$$

and I can do the sum exactly.

$$
r=r_{12}+\left[t_{12} r_{23} t_{21} e^{2 i k_{2} d}\right]\left[\frac{1}{1-r_{23} r_{21} e^{2 i k_{2} d}}\right]
$$

You can obtain for yourself with the help of Jackson page 306:

$$
r_{i j}=\frac{n_{i}-n_{j}}{n_{i}+n_{j}}=\frac{k_{i}-k_{j}}{k_{i}+k_{j}}
$$

And

$$
t_{i j}=\frac{2 n_{i}}{n_{i}+n_{j}}=\frac{2 k_{i}}{k_{i}+k_{j}}
$$

With these formulae, I'll show the following useful relationships:

$$
r_{12}=\frac{n_{1}-n_{2}}{n_{1}+n_{2}}=-\frac{n_{2}-n_{1}}{n_{1}+n_{2}}=-r_{21}
$$

And

$$
\begin{gathered}
t_{12} t_{21}=\left(\frac{2 n_{1}}{n_{1}+n_{2}}\right)\left(\frac{2 n_{1}}{n_{1}+n_{2}}\right)=\frac{4 n_{1} n_{2}}{\left(n_{1}+n_{2}\right)^{2}} \\
t_{12} t_{21}=1-\frac{\left(n_{1}-n_{2}\right)^{2}}{\left(n_{1}+n_{2}\right)^{2}}=1+r_{12} r_{21}
\end{gathered}
$$

Plug these into equation.

$$
r=r_{12}+\frac{\left(1+r_{12} r_{21}\right) e^{2 i k_{2} d}}{1+r_{12} r_{23} e^{2 i k_{2} d}}=\frac{r_{12}+r_{23} e^{2 i k_{2} d}}{1+r_{12} r_{23} e^{2 i k_{2} d}}
$$

The reflection coefficient is $R=|r|^{2}$.

$$
R=\frac{r_{12}^{2}+r_{23}^{2}+2 r_{12} r_{23} \cos \left(2 k_{2} d\right)}{1+2 r_{12} r_{23} \cos \left(2 k_{2} d\right)+\left(r_{12} r_{23}\right)^{2}}
$$

And it follows from $R+T=1$ that

$$
T=\frac{1-r_{12}^{2}-r_{23}^{2}+\left(r_{12} r_{23}\right)^{2}}{1+2 r_{12} r_{23} \cos \left(2 k_{2} d\right)+\left(r_{12} r_{23}\right)^{2}}
$$

$R+T=1$ is reasonable if we demand that energy be conserved after a long period has elapsed.
Now, here are all the crazy sketches Jackson wants:

Tu sketch this rall $k_{2}=\frac{n_{2} w}{c}$ al leet $d=C$

1. $n_{1}=1, n_{2}=2, n_{3}=3$
$r_{12}=\frac{-1}{2}, r_{23}=\frac{-1}{5}, \quad k_{2}=\frac{2}{c} w$


$R=\frac{1 / 4+1 / 5+1 / 5 \cos 4 w}{1+\frac{1}{10}+\frac{1}{5} \cos 4 w} \quad T=\frac{1-1 / 4-1 / 25+(1 / 6)^{2}}{1+\frac{1}{100}+\frac{1}{5} \cos 4 w}$
2. $n=3, n_{2}=2, \quad n_{3}=1$

$$
r_{12}=\frac{1}{5}, r_{33}=\frac{1}{2} \quad k_{2}=\frac{2}{c} w
$$

$$
\begin{aligned}
& R: \underbrace{1+\frac{1}{10}+\frac{1}{5} \cos 4 w}_{\frac{1 / 4}{1+1 / s+\frac{1}{2} \cos 4_{w}}} \\
& T=\frac{1-1 / 4-1 / 2 c \cdot(1 / 10)^{2}}{1+\frac{1}{10}+\frac{1}{s} \cos 4 \omega}
\end{aligned}
$$

3. $n_{1}=2, n_{2}=n, n_{3}=1$

$$
r_{.2}=-\frac{1}{3}, \quad r_{23}=\frac{3}{5}, \quad k_{2}=\frac{4}{c} w
$$


$R=\frac{1 / a+1 / 5-\frac{2}{5} \cos 8 \omega}{1+1 / 2-\frac{2}{5} \cos 8 \omega}$


## Problem 7.12

The time dependence of electrical disturbances in good conductors is governed by the frequency-dependent conductivity (Jackson's equation 7.58). Consider longitudinal electric fields in a conductor, using Ohm's law, the continuity equation, and the differential form of Coulomb's law.
a. Show that the time-Fourier transformed charge density satisfies the equation:

$$
\left[\sigma(\omega)-i \omega \epsilon_{0}\right] \rho(\omega)=0
$$

The continuity equation states

$$
\nabla \cdot \vec{J}_{f}=-\frac{\partial \rho_{f}}{\partial t}
$$

From Ohm's law, $\vec{J}_{f}=\sigma \vec{E}$ so

$$
\nabla \cdot \vec{J}_{f}=\nabla \cdot\left(\sigma \vec{E}_{f}\right)=\sigma \nabla \cdot \vec{E}_{f}
$$

The last step is true if $\sigma$ is uniform. According to Coulumb's law, $\nabla \cdot \vec{E}=\frac{\rho}{\epsilon_{0}}$. We now have

$$
\nabla \cdot \vec{J}=\sigma \nabla \cdot \vec{E}=\frac{\sigma}{\epsilon_{0}} \rho=-\frac{\partial \rho}{\partial t}
$$

From now on, I'll drop the subscript $f$. We both know that I mean free charge and current. From the last equality,

$$
\begin{equation*}
\sigma \rho+\epsilon_{0} \frac{\partial \rho}{\partial t}=0 \tag{1}
\end{equation*}
$$

Assume that $\rho(t)$ can be written as the time Fourier transform of $\rho(\omega)$. I.e.

$$
\rho(t)=\frac{1}{\sqrt{2 \pi}} \int \rho(\omega) e^{-i \omega t} d \omega
$$

Plug $\rho(t)$ into equation 1 .

$$
\frac{1}{\sqrt{2 \pi}} \int\left(\sigma \rho(\omega) e^{-i \omega t}+\epsilon_{0} \rho(\omega) \frac{\partial}{\partial t} e^{-i \omega t}\right) d \omega=0
$$

For the integral to vanish the integrand must vanish so

$$
\left[\sigma-i \omega \epsilon_{0}\right] \rho(\omega) e^{-i \omega t}=0
$$

For all $t$. We conclude that

$$
\left[\sigma(\omega)-i \omega \epsilon_{0}\right] \rho(\omega)=0
$$

## b. Using the representation,

$$
\sigma(\omega)=\frac{\sigma_{0}}{1-i \omega \tau}
$$

where $\sigma_{0}=\epsilon_{0} \omega_{p}^{2} \tau$ and $\tau$ is a damping time, show that in the approximation $\omega_{p} \tau \gg 1$ any initial disturbance will oscillate with the plasma frequency and decay in amplitude with a decay constant $\lambda=\frac{1}{2 \tau}$. Note that if you use $\sigma(\omega) \simeq \sigma(0)=\sigma_{0}$ in part a, you will find no oscillations and extremely rapid damping with the (wrong) decay constant $\lambda_{w}=\frac{\sigma_{0}}{\epsilon_{0}}$.
From part a, $\sigma(\omega)-i \epsilon_{0} \omega=0$. Let $\omega=-i \alpha$ so that

$$
\sigma(\omega)=\frac{\epsilon_{0} \omega_{p}^{2} \tau}{1-\alpha \tau}
$$

And the result from part a becomes $\sigma(\omega)-\epsilon_{0} \alpha=0 \rightarrow$

$$
\frac{\epsilon_{0} \omega_{p}^{2} \tau}{1-\alpha \tau}-\epsilon_{0} \alpha=0 \rightarrow \frac{\epsilon_{0} \omega_{p}^{2} \tau-\epsilon_{0} \alpha+\alpha^{2} \tau \epsilon_{0}}{1-\alpha \tau}=0
$$

The numerator must vanish. Divide the numerator by $\tau \epsilon_{0}$,

$$
\alpha^{2}-\tau^{-1} \alpha+\omega_{p}^{2}=0
$$

Solve for $\alpha$.

$$
\alpha=\frac{1}{2}\left[\tau^{-1} \pm \sqrt{\tau^{-2}-4 \omega_{p}^{2}}\right]
$$

If $\omega_{p} \gg 1$, we can write $\alpha$ in an approximate form,

$$
\alpha \sim(2 \tau)^{-1} \pm i \omega_{p}
$$

The imaginary part corresponds to the oscillations at $\omega_{p}$, the plasma frequency. The real part is the decay in amplitude $\frac{1}{2 \tau}$.

## Problem 7.16

Plane waves propagate in an homogeneous, non-permeable, but anisotropic dielectric. The dielectric is characterized by a tensor $\epsilon_{i j}$, but if coordinate axes are chosen as the principle axes, the components of displacement along these axes are related to the electric field components by $D_{i}=\epsilon_{i} E_{i}$, where $\epsilon_{i}$ are the eigenvalues of the matrix $\epsilon_{i j}$.
a. Show that plane waves with frequency $\omega$ and wave vector $\vec{k}$ must satisfy

$$
\vec{k} \times(\vec{k} \times \vec{E})+\mu_{0} \omega^{2} \vec{D}=0
$$

Consider the second Maxwell equation, $\nabla \times \vec{E}=-\frac{\partial \vec{B}}{\partial t}$. Take the curl of both sides. Plug in the fourth Maxwell equation for $\nabla \times \vec{B}$.

$$
\nabla \times(\nabla \times \vec{E})=-\nabla \times\left(\frac{\partial \vec{B}}{\partial t}\right)=-\frac{\partial}{\partial t}\left(\mu_{0} \vec{J}+\mu_{0} \frac{\partial \vec{D}}{\partial t}\right)
$$

When $\vec{J}=0$ this becomes

$$
\nabla \times(\nabla \times \vec{E})+\frac{\partial^{2}}{\partial t^{2}} \vec{D}=0
$$

Assume a solution of the form, $\vec{E}=\vec{E}_{0} e^{i(\vec{k} \cdot \vec{r}-\omega t)}$, and try it.

$$
\vec{k} \times(\vec{k} \times \vec{E})+\mu_{0} \omega^{2} \frac{\partial^{2}}{\partial t^{2}} \vec{D}=0
$$

Use $[\vec{k} \times(\vec{k} \times \vec{E})]_{i}=k_{i}(\vec{k} \cdot \vec{E})-k^{2} E_{i}=$ to write the double curl out in expanded form.

$$
\begin{equation*}
k_{i}(\vec{k} \cdot \vec{E})-k^{2} E_{i}+\mu_{0} \omega^{2} \frac{\partial^{2}}{\partial t^{2}} D_{i}=0 \tag{2}
\end{equation*}
$$

Because $D_{i}=\epsilon_{i j} E_{j}$,

$$
k_{i}(\vec{k} \cdot \vec{E})-k^{2} E_{i}+\mu_{0} \omega^{2} \frac{\partial^{2}}{\partial t^{2}} \epsilon_{i j} E_{j}=0
$$

Note $\vec{D}$ is not necessarily parallel to $\vec{E}$.
b. Show that for a given wave vector $\vec{k}=k \vec{n}$ there are two distinct modes of propagation with different phase velocities $v=\frac{\omega}{k}$ that satisfy the Fresnel equation

$$
\sum_{i=1}^{3} \frac{n_{i}^{2}}{v^{2}-v_{i}^{2}}=0
$$

where $v_{i}=\frac{1}{\sqrt{\mu_{0} \epsilon_{i}}}$ is called the principal velocity, and $n_{i}$ is the component of $\vec{n}$ along the $i$-th principal axis.
We will write the result in part a as a matrix equation. The non-diagonal elements of $\epsilon_{i j}$ vanish so we replace $\epsilon_{i j} \rightarrow \epsilon_{i i} \delta_{i j}$. Define a second rank tensor, $\overleftrightarrow{T}$, as

$$
T_{i j}=k_{i} k_{j}-\left(k^{2}-\frac{\omega_{i}^{2}}{c^{2}} \epsilon_{i i}^{2}\right) \delta_{i j}
$$

The result in equation 2 can be written $\overleftrightarrow{T} \cdot \vec{E}=0$. In order for there to be a nontrivial solution det $\overleftrightarrow{T}=0$. Divide $\overleftrightarrow{T}$ by $k^{2}$ and use $\frac{k_{i}}{|k|}=n_{i}$ to make things look cleaner

$$
T_{i j} / k^{2}=n_{i} n_{j}-\left(1-\frac{\omega^{2}}{k^{2} c^{2}} \epsilon_{i i}^{2}\right) \delta_{i j}
$$

We remember the relations $v=\frac{\omega}{k}$ and $v_{i}=\frac{c}{\sqrt{\epsilon i}}$. So we have

$$
T_{i j} / k^{2}=n_{i} n_{j}-\left(1-\frac{v^{2}}{v_{i}^{2}}\right) \delta_{i j}
$$

At this point, we can solve det $\overleftrightarrow{T}=0$ for the allowed velocity values.

$$
\operatorname{det}\left(\begin{array}{ccc}
n_{1}^{2}-\left(1-\frac{v^{2}}{v_{1}^{2}}\right) & n_{2} n_{1} & n_{3} n_{1} \\
n_{1} n_{2} & n_{2}^{2}-\left(1-\frac{v^{2}}{v_{2}^{2}}\right) & n_{3} n_{2} \\
n_{1} n_{3} & n_{2} n_{3} & n_{3}^{2}-\left(1-\frac{v^{2}}{v_{3}^{2}}\right)
\end{array}\right)=0
$$

Or written out explicitly, we have

$$
\begin{array}{r}
\left(\frac{v^{2}}{v_{1}^{2}}-1\right)\left(\frac{v^{2}}{v_{2}^{2}}-1\right)\left(\frac{v^{2}}{v_{3}^{2}}-1\right) \\
+n_{1}^{2}\left(\frac{v^{2}}{v_{1}^{2}}-1\right)\left(\frac{v^{2}}{v_{2}^{2}}-1\right)\left(\frac{v^{2}}{v_{3}^{2}}-1\right)
\end{array}
$$

$$
\begin{array}{r}
+n_{2}^{2}\left(\frac{v^{2}}{v_{1}^{2}}-1\right)\left(\frac{v^{2}}{v_{2}^{2}}-1\right)\left(\frac{v^{2}}{v_{3}^{2}}-1\right) \\
+n_{3}^{2}\left(\frac{v^{2}}{v_{1}^{2}}-1\right)\left(\frac{v^{2}}{v_{2}^{2}}-1\right)\left(\frac{v^{2}}{v_{3}^{2}}-1\right)=0
\end{array}
$$

Multiplying out the determinant,

$$
\begin{aligned}
& \left(\frac{v^{2}}{v_{1}^{2}}-1\right)\left(\frac{v^{2}}{v_{2}^{2}}-1\right)\left(\frac{v^{2}}{v_{3}^{2}}-1\right)+n_{1}^{2}\left(\frac{v^{2}}{v_{2}^{2}}-1\right)\left(\frac{v^{2}}{v_{3}^{2}}-1\right) \\
& +n_{2}^{2}\left(\frac{v^{2}}{v_{3}^{2}}-1\right)\left(\frac{v^{2}}{v_{1}^{2}}-1\right)+n_{3}^{2}\left(\frac{v^{2}}{v_{1}^{2}}-1\right)\left(\frac{v^{2}}{v_{2}^{2}}-1\right)=0
\end{aligned}
$$

which can be written in a nicer form.

$$
1+\frac{n_{1}^{2} v_{1}^{2}}{v^{2}-v_{1}^{2}}+\frac{n_{2}^{2} v_{2}^{2}}{v^{2}-v_{2}^{2}}+\frac{n_{3}^{2} v_{3}^{2}}{v^{2}-v_{3}^{2}}=0
$$

Use $n_{1}^{2}+n_{2}^{2}+n_{3}^{3}=1$ to replace the number one in the above equation.

$$
\begin{array}{r}
\frac{n_{1}^{2}\left(v^{2}-v_{1}^{2}\right)}{v^{2}-v_{1}^{2}}+\frac{n_{2}^{2}\left(v^{2}-v_{2}^{2}\right)}{v^{2}-v_{2}^{2}}+\frac{n_{3}^{2}\left(v^{2}-v_{3}^{2}\right)}{v^{2}-v_{3}^{2}} \\
\quad+\frac{n_{1}^{2} v_{1}^{2}}{v^{2}-v_{1}^{2}}+\frac{n_{2}^{2} v_{2}^{2}}{v^{2}-v_{2}^{2}}+\frac{n_{3}^{2} v_{3}^{2}}{v^{2}-v_{3}^{2}}=0
\end{array}
$$

Simplify. In the end, you'll obtain a relationship for the $v$ values.

$$
n_{3}^{2}\left(v^{2}-v_{1}^{2}\right)\left(v^{2}-v_{2}^{2}\right)+n_{1}^{2}\left(v^{2}-v_{3}^{2}\right)\left(v^{2}-v_{2}^{2}\right)+n_{2}^{2}\left(v^{2}-v_{3}^{2}\right)\left(v^{2}-v_{1}^{2}\right)=0
$$

This is quadratic in $v^{2}$ so we expect two solutions for $v^{2}$. Divide by $\left(v^{2}-\right.$ $\left.v_{1}^{2}\right)\left(v^{2}-v_{2}^{2}\right)\left(v^{2}-v_{3}^{2}\right)$ and write in the compact form which Jackson likes:

$$
\sum_{i=1}^{3} \frac{n_{i}^{2}}{v^{2}-v_{i}^{2}}=0
$$

c. Show that $\vec{D}_{a} \cdot \vec{D}_{b}=0$, where $\vec{D}_{a}, \vec{D}_{b}$ are displacements associated with two modes of propagation.
Divide the equation 2 by $k^{2}$ to find the equations which the eigenvectors must satisfy:

$$
\begin{equation*}
\vec{E}_{1}-\vec{n}\left(\vec{n} \cdot \vec{E}_{1}\right)=\frac{v_{1}^{2}}{c^{2}} \vec{D}_{1} \tag{3}
\end{equation*}
$$

And

$$
\begin{equation*}
\vec{E}_{2}-\vec{n}\left(\vec{n} \cdot \vec{E}_{2}\right)=\frac{v_{2}^{2}}{c^{2}} \vec{D}_{2} \tag{4}
\end{equation*}
$$

Dot the first equation by $E_{2}$ and the second by $E_{1}$.

$$
\vec{E}_{2} \cdot \vec{E}_{1}-\left(\vec{E}_{2} \cdot \vec{n}\right)\left(\vec{n} \cdot \vec{E}_{1}\right)=\frac{v_{1}^{2}}{c^{2}} \vec{E}_{2} \cdot \vec{D}_{1}
$$

And

$$
\vec{E}_{2} \cdot \vec{E}_{1}-\left(\vec{E}_{2} \cdot \vec{n}\right)\left(\vec{n} \cdot \vec{E}_{1}\right)=\frac{v_{2}^{2}}{c^{2}} \vec{E}_{1} \cdot \vec{D}_{2}
$$

Comparing these, we see that

$$
v_{1}^{2} \vec{E}_{2} \cdot \vec{D}_{1}=v_{2}^{2} \vec{E}_{1} \cdot \vec{D}_{2}
$$

Well, we already know that in general $v_{1} \neq v_{2}$. So $\vec{E}_{2} \cdot \vec{D}_{1}$ and $\vec{E}_{1} \cdot \vec{D}_{2}$ must either vanish or be related in such a way as to preserve the equality. However, $\vec{E}_{2} \cdot \vec{D}_{1}=\vec{E}_{1} \cdot \vec{D}_{2}$ because $\epsilon_{i j}$ is diagonal. Then, $\epsilon_{i j} E_{1 i} E_{2 j} \delta_{i j}=\epsilon_{i j} E_{2 j} E_{1 i} \delta_{i j}$. Therefore, we must conclude that $\vec{E}_{2} \cdot \vec{D}_{1}=\vec{E}_{1} \cdot \vec{D}_{2}=0$.
Dot product equation 3 into 4 and find

$$
\vec{E}_{2} \cdot \vec{E}_{1}+\left(\hat{n} \cdot \vec{E}_{1}\right)\left(\hat{n} \cdot \vec{E}_{2}\right)-2\left(\hat{n} \cdot \vec{E}_{1}\right)\left(\hat{n} \cdot \vec{E}_{2}\right)=\frac{v_{1}^{2} v_{2}^{2}}{c^{4}} \vec{D}_{2} \cdot \vec{D}_{1}
$$

The left hand side can be rewritten as $\vec{E}_{2} \cdot \vec{E}_{1}-\left(\vec{E}_{2} \cdot \vec{n}\right)\left(\vec{n} \cdot \vec{E}_{1}\right) \propto \vec{E}_{1} \cdot \vec{D}_{2}$ which we have shown to vanish. Therefore, the left hand side is zero, and the right hand side, $\vec{D}_{1} \cdot \vec{D}_{2}=0$. The eigenvectors are perpendicular.

## Problem 7.22

Use the Kramers-Krönig relation to calculate the real part of $\epsilon(\omega)$, given the imaginary part of $\epsilon(\omega)$ for positive $\omega$ as . . .
The Kramer-Krönig relation states:

$$
\Re\left(\frac{\epsilon(\omega)}{\epsilon_{0}}\right)=1+\frac{2}{\pi} P \int_{0}^{\infty} \frac{\omega^{\prime}}{\omega^{\prime 2}-\omega^{2}} \Im\left(\frac{\epsilon\left(\omega^{\prime}\right)}{\epsilon_{0}}\right) d \omega^{\prime}
$$

a. $\Im\left(\frac{\epsilon(\omega)}{\epsilon_{0}}\right)=\lambda\left[\Theta\left(\omega-\omega_{1}\right)-\Theta\left(\omega-\omega_{2}\right)\right]$.

Plug this into the Kramer-Kronig relationship.

$$
\Re\left(\frac{\epsilon(\omega)}{\epsilon_{0}}\right)=1+\frac{2 \lambda}{\pi} \int_{\omega_{1}}^{\omega_{2}} \frac{\omega^{\prime}}{\omega^{\prime 2}-\omega^{2}} d \omega^{\prime}+0
$$

Notice that the real part of $\epsilon(\omega)$ depends on an integral over the entire frequency range for the imaginary part!
Here, we will use a clever trick.

$$
\Re\left(\frac{\epsilon(\omega)}{\epsilon_{0}}\right)=1+\frac{\lambda}{\pi} \int_{\omega_{1}^{2}}^{\omega_{2}^{2}} \frac{d\left(\omega^{\prime 2}\right)}{\omega^{\prime 2}-\omega^{2}}
$$

And this integral is easy to do!

$$
\Re\left(\frac{\epsilon(\omega)}{\epsilon_{0}}\right)=1+\left.\frac{\lambda}{\pi} \ln \left(\omega^{\prime 2}-\omega^{2}\right)\right|_{\omega_{1}^{2}} ^{\omega_{2}^{2}}=1+\frac{\lambda}{\pi} \ln \left(\frac{\omega_{2}^{2}-\omega^{2}}{\omega_{1}^{2}-\omega^{2}}\right)
$$

b. $\Im\left(\frac{\epsilon(\omega)}{\epsilon_{0}}\right)=\frac{\lambda \gamma \omega}{\left(\omega_{0}^{2}-\omega^{2}\right)+\gamma^{2} \omega^{2}}$

Do the same thing.

$$
\Re\left(\frac{\epsilon(\omega)}{\epsilon_{0}}\right)=1+\frac{2}{\pi} P \int_{0}^{\infty} \frac{\lambda \gamma \omega^{\prime 2}}{\left(\left(\omega_{0}^{2}-\omega^{2}\right)^{2}+\gamma^{2} \omega^{2}\right)\left(\omega^{\prime 2}-\omega^{2}\right)} d \omega^{\prime}
$$

The integral can be evaluated using complex analysis, but I'll avoid this. The integral is just a Hilbert transformation and you can look it up in a table.

$$
\Re\left(\frac{\epsilon(\omega)}{\epsilon_{0}}\right)=1+\frac{\lambda\left(\omega_{0}^{2}-\omega^{2}\right)}{\left(\omega_{0}^{2}-\omega^{2}\right)^{2}+\omega^{2} \gamma^{2}}
$$



## Problem 9.3

## A radiating

$V(t)=V_{0} \cos (\omega t)$. There should be a diagram showing a sphere split across the equator. The top half is kept at a potential $V(t)$ while the bottom half is $-V(t)$. From Jackson 9.9,

$$
\lim _{k r \rightarrow \infty} \vec{A}(\vec{x})=\frac{\mu_{0}}{4 \pi} \frac{e^{i k r}}{r} \sum_{n} \frac{(-i k)^{n}}{n!} \int \vec{J}\left(\vec{x}^{\prime}\right)\left(n \cdot \vec{x}^{\prime}\right)^{n} d V^{\prime}
$$

If $k d=k 2 R \ll 1$ (as it is), the higher order terms in this expansion fall off rapidly. In our case, it is sufficient to consider just the first term.

$$
\vec{A}(\vec{x})=\frac{\mu_{0}}{4 \pi} \frac{e^{i k r}}{r} \int \vec{J}\left(\vec{x}^{\prime}\right) d V^{\prime}
$$

Integrating by parts and substituting $\nabla \cdot \vec{J}=i \omega \rho$, we find

$$
\vec{A}=\frac{-i \mu_{0} \omega}{4 \pi} \vec{p} \frac{e^{i k r}}{r}
$$

I solved the static situation (but neglected to include it) earlier.

$$
\Phi=V\left[\frac{3}{2}\left(\frac{R}{r}\right)^{2} P_{1}(\cos \theta)-\frac{7}{8}\left(\frac{R}{r}\right)^{4} P_{3}(\cos \theta)+\frac{11}{16}\left(\frac{R}{r}\right)^{6} P_{5}(\cos \theta)+\ldots\right]
$$

Written a different way, this is

$$
\Phi=\frac{1}{4 \pi \epsilon_{0}} \frac{Q}{r}+\frac{1}{4 \pi \epsilon_{0}} \frac{\vec{p} \cdot \vec{x}}{r^{3}}+\ldots
$$

Choose the $z$-axis so that $\vec{p} \cdot \vec{x}=p r \cos \theta$. Compare like terms between the two expressions for $\Phi$ to find the dipole moment in terms of known variables.

$$
V \frac{3}{2}\left(\frac{R}{r}\right)^{2} \cos \theta=\frac{1}{4 \pi \epsilon_{0}} \frac{p r \cos \theta}{r^{3}}
$$

So

$$
\left|p_{0}\right|=\frac{3}{2} 4 \pi \epsilon_{0} V_{0} R^{2}
$$

The time dependent dipole moment is

$$
\vec{p}(t)=6 \pi \epsilon_{0} V_{0} R^{2} \cos (\omega t) \hat{z}
$$

And with this,

$$
\vec{A}=\frac{-i \mu_{0} \omega}{4 \pi} \frac{e^{i k r}}{r} p_{0} \hat{z}
$$

In the radiation zone, Jackson said that

$$
\vec{H}=\frac{c k^{2}}{4 \pi}(\vec{n} \times \vec{p}) \frac{e^{i k r}}{r}
$$

And

$$
\vec{E}=\sqrt{\frac{\mu_{0}}{\epsilon_{0}}} \vec{H} \times \hat{n}
$$

Some texts will use $z_{0}=\sqrt{\frac{\mu_{0}}{\epsilon_{0}}} ;$ I don't. First, find the magnetic field. Use $\omega=k c$.

$$
\vec{B}=\mu_{0} \vec{H}=-\frac{\mu_{0} p_{0} \omega^{2}}{4 \pi c}\left(\frac{\sin \theta}{r}\right) e^{i k r} \hat{\phi}
$$

Then, find the electric field.

$$
\vec{E}=\frac{\mu_{0} p_{0} \omega^{2}}{4 \pi}\left(\frac{\sin \theta}{r}\right) e^{i k r} \hat{\theta}
$$

The power radiated per solid angle can be obtained from the Poynting vector.

$$
\frac{d P}{d \Omega}=\frac{r^{2}}{2 \mu_{0}}\left|\vec{E}^{*} \times \vec{B}\right|=\frac{\mu_{0}}{2 c}\left[\frac{p_{0}^{2} \omega^{2}}{16 \pi^{2}} \sin ^{2} \theta\right] \hat{r}
$$

Notice how the complex conjugation and absolute signs get rid of the pesky wave factors.
Integrate over all solid angles to find the total radiated power.

$$
P_{\text {Total }}=\int \frac{\mu_{0}}{2 c}\left[\frac{p_{0}^{2} \omega^{2}}{16 \pi^{2}} \sin ^{2} \theta\right] d \Omega=\frac{\mu_{0} p_{0}^{2} \omega^{4}}{16 \pi c} \int_{0}^{\pi} \sin ^{3} \theta d \theta
$$

The final integral is quite simple, but I'll solve it anyway.

$$
\int_{0}^{\pi} \sin ^{3} \theta d \theta=\left.\frac{-1}{3} \cos \theta\left(\sin ^{2} \theta+2\right)\right|_{\theta=0} ^{\pi}
$$

Putting all this together, the final result is

$$
P_{\text {Total }}=\frac{3 \pi \epsilon_{0} V_{0}^{2} R^{4} \omega^{4}}{c^{3}}
$$

## Problem 9.10

The transitional charge and current densities for the radiative transition from the excited state $m=0,2 p$ in hydrogen to the ground state $1 s$, i.e.

$$
|H ; 2 p\rangle \rightarrow|1 s\rangle
$$

are, in the notation of (9.1) and with the neglect of spin, the matrix elements, $\langle 2 p| \hat{\rho}|1 s\rangle$ :

$$
\rho(r, \theta, \phi, t)=\frac{2 q}{\sqrt{b} a_{0}^{4}} r e^{-\frac{3 r}{2 a_{0}}} Y_{00} Y_{10} e^{-i \omega t}
$$

We also know the current density.

$$
\vec{J}(r, \theta, \phi, t)=-i v_{0}\left(\frac{\hat{r}}{2}+\frac{a_{0}}{z} \hat{z}\right) \rho(r, \theta, \phi, t)
$$

where $a_{0}=\frac{4 \pi \epsilon_{0} \hbar^{2}}{m e^{2}}=0.529 \times 10^{-19} \mathrm{~m}$ is the Bohr radius, $\omega_{0}=\frac{3^{2}}{32 \pi \epsilon_{0} \hbar a_{0}}$ is the frequency difference of the levels, and $v_{0}=\frac{e^{2}}{4 \pi \epsilon_{0} \hbar}=\alpha c \approx \frac{c}{137}$ is the Bohr orbit speed.
a. Find the effective transitional "magnetization", calculate $\nabla \cdot \vec{M}$, and evaluate all the non-vanishing radiation multi-poles in the longwavelength limit.
The magnetization is

$$
\vec{M}=\frac{1}{2}(\vec{r} \times \vec{J})
$$

$\vec{J}$ can be broken up into $J_{r}$ and $J_{z}$ components. We take the cross product of the two components with $\vec{r}$.

$$
\vec{r} \times J_{r}=0
$$

And

$$
\vec{r} \times J_{z}=-i v_{0}\left(\frac{-x}{z} \hat{y}+\frac{y}{z} \hat{x}\right) a_{0} \rho
$$

To make things easier, we'll use angles. $\tan \theta=\frac{r}{z}, \sin \phi=\frac{y}{r}, \cos \phi=\frac{x}{r}$ Then,

$$
\vec{r} \times \vec{J}=-i a_{0} \rho v_{0}(\tan \theta \sin \phi \hat{x}-\tan \theta \cos \phi \hat{y})
$$

Don't forget $v_{0}=\alpha c$, so

$$
\vec{M}=-i \frac{\alpha c a_{0}}{2} \tan \theta(\sin \phi \hat{x}-\cos \phi \hat{y}) \rho
$$

Let $\vec{\chi}=\frac{-i \alpha c a_{0}}{2} \tan \theta(\sin \phi \hat{x}-\cos \phi \hat{y})$ then

$$
\vec{M}=\rho \vec{\chi}
$$

Now, we take the divergence.

$$
\nabla \cdot \vec{M}=(\nabla \cdot \vec{\chi}) \rho+\vec{\chi} \cdot \nabla \rho
$$

We'll consider each term separately to show that they all vanish. First of all,

$$
\nabla \cdot \chi \sim \nabla \cdot\left(\frac{y}{z} \hat{x}-\frac{x}{z} \hat{y}\right)=0
$$

Now since $\rho \sim r e^{\frac{-3 r}{2 a_{0}}} \cos \theta=z e^{\frac{-3 r}{2 a_{0}}}$, its gradient is

$$
\nabla \rho=z e^{\frac{-3 r}{2 a_{0}}}\left[\frac{-3 x}{2 a_{0} r} \hat{x}+\frac{-3 y}{2 a_{0} r} \hat{y}+\left(\frac{1}{z}-\frac{3 z}{2 a_{2} r}\right) \hat{z}\right]
$$

Which is orthogonal to $\chi$

$$
\chi \cdot \nabla \rho \sim \frac{y x}{r}-\frac{x y}{r}=0
$$

Both terms in the divergence vanish, and $\nabla \vec{M}=0$.
The dipole moment is

$$
\vec{p}=\int(x \hat{x}+y \hat{y}+z \hat{z}) \rho(\vec{x}) d V
$$

Don't forget

$$
\rho(\vec{x})=\kappa z e^{\frac{-3}{2 a_{0}} \sqrt{x^{2}+y^{2}+z^{2}}}
$$

where $\kappa=\frac{2 q}{\sqrt{6} a_{0}^{4}} \frac{1}{\sqrt{4 \pi}} \sqrt{\frac{3}{4 \pi}}$. Putting this together,

$$
\vec{p}=\int z(x \hat{x}+y \hat{y}+z \hat{z}) \kappa e^{\frac{-3 r}{2 a_{0}}} d V
$$

Obviously,

$$
\int_{-\infty}^{\infty} u e^{-f(u)} d u=0
$$

if $f(u)$ is even. Thus, the integrals over the $x$ and $y$ coordinates vanish. We are left with

$$
\vec{p}=\kappa \hat{z} \int_{-\infty}^{\infty} z^{2} e^{\frac{-3 r}{2 a_{0}}} d V=2 \pi \kappa \hat{z}\left(\int r^{4} e^{-3 r} 2 a_{0} d r\right)\left(\int \cos ^{2} \theta d(\cos \theta)\right)
$$

Use

$$
\int_{0}^{\infty} r^{n} e^{-\beta r} d r=\frac{n!}{\beta^{n+1}} \rightarrow \int_{0}^{\infty} r^{4} e^{-\beta r} d r=\frac{4!}{\beta^{5}}
$$

And

$$
\int_{-1}^{1} \cos ^{2} \theta d(\cos \theta)=\frac{2}{3}
$$

To get

$$
\vec{p}=\kappa \hat{z}\left(\frac{24}{3^{5}} 2^{5} a_{0}^{5}\right)\left(\frac{2}{3}\right)(2 \pi)
$$

Plug in $\kappa$ explicitly.

$$
\vec{p}=1.49 q a_{0} \hat{z}
$$

Now, for the magnetic moment,

$$
\vec{m}=\int \vec{M} d V=\frac{-i a_{0} v_{0}}{2} \int \rho\left(\frac{y}{z} \hat{x}-\frac{x}{z} \hat{y}\right)
$$

Well, $\rho$ is even but $y$ and $x$ are odd so $\vec{m}$ is zero. The magnetic dipole and electric quadrapole terms vanish because of their dependence on $m$.
We suspect that electric octo-pole and every other pole thereafter might persist because of symmetry, but we won't worry about that.
b. In the electric dipole approximation calculate the total timeaveraged power radiated. Express your answer in units of $\left(\hbar \omega_{0}\right)\left(\frac{\alpha^{4} c}{a_{0}}\right)$.

$$
P=\frac{c^{2} z_{0} k^{4}}{12 \pi} \vec{p}^{2}
$$

where $z_{0}=\frac{1}{\epsilon c}$. Now, $\hbar \alpha=\frac{q^{2}}{4 \pi \epsilon_{0} c}$. With some fiddling,

$$
P_{J a c k s o n}=3.9 \times 10^{-2}\left(\hbar \omega_{0}\right)\left(\frac{\alpha^{4} c}{a_{0}}\right)
$$

c. Interpreting the classically calculated power as the photon energy times the transition probability, evaluate numerically the transition probability in units of reciprocal seconds.
$\hbar \omega \Gamma=P$. Using numbers, $\Gamma=6.3 \times 10^{8}$ seconds $^{-1}$.
d. If, instead of the semi-classical charge density used above, the electron in the $2 p$ state was described by a circular Bohr orbit of radius $2 a_{0}$, rotating with the transitional frequency $\omega_{0}$, what would the predicted power be? Express your answer in the same units as in part $b$ and evaluate the ratio of the two powers numerically.
For a Bohr transition, a dipole transition,

$$
\vec{p}=q\left(2 a_{0}-a_{0}\right) \hat{z} e^{-i \omega t}=q a_{0} e^{-i \omega t} \hat{z}
$$

which gives an emitted power of $P_{B o h r}=0.018\left(\hbar \omega_{0}\right)\left(\frac{\alpha^{4} c}{a_{0}}\right)$. And the ratio:

$$
\frac{P_{\text {Bohr }}}{P_{\text {Jackson }}} \simeq 0.45
$$

The grader claims that this is incorrect citing a correct value of 0.55 . You decide, and tell me what you conclude.


Figure 1:

## Problem 9.16

A thin linear antenna of length $d$ is excited in such a way that the sinusoidal current makes a full wavelength of oscillation as shown in some Jacksonian figure.
a. Calculate exactly the power radiated per unit solid angle and plot the angular distribution of radiation.
Assume the antenna is center fed.

$$
\vec{J}=I_{0} \sin \left(\frac{1}{2} k d-k|z|\right) \delta(x) \delta(y) e^{-i \omega t} \hat{z},|z| \leq \frac{1}{2} d
$$

Note $J\left( \pm \frac{d}{2}\right)=0$ as makes sense. Jackson makes some arguments to justify this current density for a center fed antenna. I'll take his word for it, but if you're not convinced, consult Jackson page 416 in the third edition.
The vector potential due to an oscillating current is

$$
\vec{A}(\vec{r})=\frac{\mu_{0}}{4 \pi} \int \vec{J}(\vec{r}, t) \frac{e^{i k\left|\vec{r}-\vec{r}^{\prime}\right|}}{|\vec{r}-\vec{r}|} d^{3} \vec{r}^{\prime}
$$

In the radiation zone,

$$
\frac{e^{i k\left|\vec{r}-\vec{r}^{\prime}\right|}}{\left|\vec{r}-\vec{r}^{\prime}\right|} \rightarrow \frac{e^{i k r}}{r} e^{-i k \frac{\vec{r} \cdot \vec{r}^{\prime}}{r}}
$$

The vector potential with the current density can be explicitly written

$$
\vec{A}(\vec{r})=\frac{\mu_{0}}{4 \pi} \frac{e^{i k r}}{r} \hat{z} \int_{-\frac{d}{2}}^{\frac{d}{2}} I_{0} \sin \left(\frac{1}{2} k d-k|z|\right) e^{-i k z^{\prime} \cos \theta} d z^{\prime} \hat{z}
$$

Bear in mind that this expression for the vector potential includes all multipoles. The integral can be done quite easily. Use Euler's theorem, $e^{-i x}=$ $\cos x-i \sin x \rightarrow \sin x=\frac{e^{x}-e^{-x}}{2 i}$, to write:

$$
\mathcal{I}_{1}=\int \frac{1}{2 i}\left(e^{i \frac{i}{2} k d-i k\left|z^{\prime}\right|}-e^{-i \frac{1}{2} k d+i k|z|}\right) e^{-i k z^{\prime} \cos \theta} d z^{\prime} \hat{z}
$$

Written out in full,

$$
\begin{aligned}
\mathcal{I}_{1} & =\frac{1}{2 i} e^{i \frac{1}{2} k d} \int_{0}^{\frac{d}{2}} e^{(-i k-i k \cos \theta) z^{\prime}} d z^{\prime}-\frac{1}{2 i} e^{-i \frac{1}{2} k d} \int_{0}^{\frac{d}{2}} e^{(i k-i k \cos \theta) z^{\prime}} d z^{\prime} \\
& +\frac{1}{2 i} e^{i \frac{1}{2} k d} \int_{-\frac{d}{2}}^{0} e^{(i k-i k \cos \theta) z^{\prime}} d z^{\prime}-\frac{1}{2 i} e^{-i \frac{1}{2} k d} \int_{-\frac{d}{2}}^{0} e^{(-i k-i k \cos \theta) z^{\prime}} d z^{\prime}
\end{aligned}
$$

Each integral can be solved quite easily by "u " substitution.

$$
\begin{array}{r}
\mathcal{I}_{1}=\frac{1}{2 i} e^{i \frac{1}{2} k d} \frac{1-e^{(-i k-i k \cos \theta) \frac{d}{2}}}{(k+i k \cos \theta)}+\frac{1}{2 i} e^{i \frac{1}{2} k d} \frac{1-e^{(-k+i k \cos \theta) \frac{d}{2}}}{(i k-i k \cos \theta)} \\
\quad+\frac{1}{2 i} e^{-i \frac{1}{2} k d} \frac{1-e^{(-i k+i k \cos \theta) \frac{d}{2}}}{(i k-i k \cos \theta)}+\frac{1}{2 i} e^{-i \frac{1}{2} k d} \frac{1-e^{(i k+i k \cos \theta) \frac{d}{2}}}{(i k+i k \cos \theta)}
\end{array}
$$

The result is a mess. Use Maple or have patience. It takes a bit of algebra to get the neat result,

$$
I_{1}=\frac{2}{k}\left[\frac{\cos \left(\frac{1}{2} k d \cos \theta\right)-\frac{1}{2} \cos (k d)}{\sin ^{2} \theta}\right]
$$

And then,

$$
\vec{A}(\vec{r})=\frac{2 \mu_{0}}{4 \pi} \frac{e^{i k r}}{k r}\left[\frac{\cos \left(\frac{1}{2} k d \cos \theta\right)-\frac{1}{2} \cos (k d)}{\sin ^{2} \theta}\right] \hat{z}
$$

Who cares about the vector potential? We want $E$ and $B$ fields. Fortunately, we know how to write the $E$ and $B$ fields in the radiation zone in terms of the vector potential.

$$
\begin{gathered}
\vec{B}=i k \hat{r} \times \vec{A} \rightarrow\left|\vec{B}_{0}\right|=k \sin \theta\left|\overrightarrow{A_{0}}\right| \hat{\phi} \\
\vec{E}=i c k(\hat{r} \times \vec{A}) \times \hat{r} \rightarrow\left|\vec{E}_{0}\right|=c k \sin \theta\left|\overrightarrow{A_{0}}\right| \hat{\theta}
\end{gathered}
$$

The time averaged angular distribution of power is

$$
\begin{gathered}
\frac{d P}{d \Omega}=\frac{r^{2}}{2 \mu_{0}}\left|\vec{E} \times \vec{B}^{*}\right|=\frac{1}{2 \mu_{0}} c k^{2} \frac{I_{0}^{2}}{k^{2}}\left(\frac{2 \mu_{0}}{4 \pi}\right)^{2}\left[\frac{\cos \left(\frac{1}{2} k d \cos \theta\right)-\frac{1}{2} \cos k d}{\sin \theta}\right]^{2} \\
\frac{d P}{d \Omega}=\frac{2 \mu_{0} I_{0}^{2} c}{16 \pi^{2}}\left[\frac{\cos \left(\frac{1}{2} k d \cos \theta\right)-\frac{1}{2} \cos k d}{\sin \theta}\right]^{2}
\end{gathered}
$$

In this problem $\lambda=d$ so $k d=\frac{2 \pi}{\lambda} d=2 \pi$.

$$
\frac{d P}{d \Omega}=\frac{2 \mu_{0} I^{2} c}{16 \pi^{2}}\left[\frac{\cos (\pi \cos \theta)-\frac{1}{2} \cos \pi}{\sin \theta}\right]^{2}
$$

Well, $\cos \pi=-1$, and of course, $\cos \alpha+\frac{1}{2}=2 \cos ^{2}\left(\frac{\alpha}{2}\right)$.

$$
\frac{d P}{d \Omega}=\frac{8 \mu_{0} I^{2} c}{16 \pi^{2}}\left[\frac{\cos ^{4}\left(\frac{1}{2} \pi \cos \theta\right)}{\sin ^{2} \theta}\right]
$$

b. Determine the total power radiated and find a numerical value for the radiation resistance.
Integrate the result from part a over all solid angles.

$$
P_{\text {total }}=\int \frac{d P}{d \Omega} d \Omega=\frac{\mu_{0} I^{2} c}{2 \pi^{2}} \int\left[\frac{\cos ^{4}\left(\frac{1}{2} \pi \cos \theta\right)}{\sin ^{2} \theta}\right] d \Omega
$$

Integrating over $\phi$,

$$
P_{\text {total }}=\frac{\mu_{0} I^{2} c}{\pi} \int\left[\frac{\cos ^{4}\left(\frac{1}{2} \pi \cos \theta\right)}{\sin ^{2} \theta}\right] \sin \theta d \theta
$$

Obviously $^{3}$, the integral equals about 0.84 .

$$
P=(0.84) \frac{I_{0}^{2} \mu_{0} c}{\pi}
$$

We learned in high school that $P=I^{2} R$ and it does take much to show $R=\frac{P}{I^{2}}=\frac{\mu_{0} c}{8 \pi}(6.7) \approx 100 \Omega$. Here, $\Omega$ stands for Ohms. Actually, Jackson seems to define the radiative resistance as 2 times this, but typically Jackson is hard to follow so I'll ignore this factor without a better explanation about its origin.

[^2]
## Problem 9.17

Treat the linear antenna of Problem 9.16 by the multi-pole expansion method.
Until further notice: the units in this problem are inconsistent. check them! a. Calculate the multi-pole moments (electric dipole, magnetic dipole, and electric quadrupole) exactly and in the long wavelength approximation.
For a linear antenna:

$$
\vec{J}(\vec{r})=\hat{z} \sin (k z) \delta(x) \delta(y) I_{0}
$$

Use the multi-pole expansion.

$$
\lim _{k r \rightarrow \infty} \vec{A}(\vec{x})=\frac{\mu_{0}}{4 \pi} \frac{e^{i k r}}{r} \sum_{n} \frac{(-i k)^{n}}{n!} \int \vec{J}\left(\vec{x}^{\prime}\right)\left(n \cdot \vec{x}^{\prime}\right)^{n} d V^{\prime}
$$

For $\mathrm{n}=1$ in the expansion, we find the electric dipole contribution:

$$
\vec{A}=\frac{\mu_{0}}{4 \pi} \frac{e^{i k r}}{r} \int \vec{J}\left(\vec{r}^{\prime}\right) d V^{\prime}=\frac{\mu_{0}}{4 \pi} \frac{e^{i k r}}{r} \hat{z} I_{0} \int_{-\frac{d}{2}}^{\frac{d}{2}} \sin \left(k z^{\prime}\right) d z^{\prime}=0
$$

When $\mathrm{n}=2$ in the expansion, we get a term proportional to the integral of $\vec{J}\left(\vec{n} \cdot \vec{r}^{\prime}\right)$. Using the vector identities, this can be rewritten in terms of the magnetic dipole and electric quadrapole contributions. The magnetic dipole contribution is:

$$
\vec{A}=-\frac{\mu_{0}}{4 \pi} \frac{e^{i k r}}{r} \frac{i k}{2} \int\left(\vec{r}^{\prime} \times \vec{J}\left(\vec{r}^{\prime}\right)\right) \times \vec{n} d V^{\prime}=0
$$

The electric quadrapole contribution is:

$$
\begin{array}{r}
\vec{A}=-\frac{\mu_{0}}{4 \pi} \frac{e^{i k r}}{r} \frac{i k}{2} \int\left[\left(\vec{n} \cdot \vec{r}^{\prime}\right) \vec{J}\left(\vec{r}^{\prime}\right)+\left(\vec{n} \cdot \vec{J}\left(\vec{r}^{\prime}\right)\right) \vec{r}^{\prime}\right] d V^{\prime} \\
=-\frac{\mu_{0}}{4 \pi} \frac{e^{i k r}}{r} \frac{i k}{2} \hat{z} I_{0} \int_{\frac{d}{2}}^{\frac{d}{2}}\left[z^{\prime} \cos \theta \sin \left(k z^{\prime}\right)+\cos \theta \sin \left(k z^{\prime}\right) z^{\prime}\right] d z^{\prime} \\
=-\frac{\mu_{0}}{4 \pi} \frac{e^{i k r}}{r} i k \hat{z} I_{0} \cos \theta \int z^{\prime} \sin \left(k z^{\prime}\right) d z^{\prime}=\frac{\mu_{0}}{4 \pi} \frac{e^{i k r}}{r} i k \hat{z} I_{0} \cos \theta \frac{\partial}{\partial k} \int \cos \left(k z^{\prime}\right) d z^{\prime} \\
=\frac{\mu_{0}}{4 \pi} \frac{e^{i k r}}{r} 2 i k \hat{z} I_{0} \cos \theta \frac{\partial}{\partial k}\left(\frac{\sin \left(\frac{k d}{2}\right)}{k}\right) \\
=\frac{\mu_{0}}{4 \pi} \frac{e^{i k r}}{r} 2 i k \hat{z} I_{0} \cos \theta\left(\frac{d}{2 k} \cos \left(\frac{k d}{2}\right)-\frac{1}{k^{2}} \sin \left(\frac{k d}{2}\right)\right)
\end{array}
$$

b. Compare the shape of the angular distribution of radiated power for the lowest non-vanishing multi-pole with the exact distribution of Problem 9.16.
We were given $k d=2 \pi$ so

$$
\vec{A}=-\frac{\mu_{0}}{4 \pi} i d \frac{e^{i k r}}{r} \hat{z} I_{0} \cos \theta
$$

And the power per solid angle

$$
\frac{d P}{d \Omega}=\frac{r^{2}}{2 \mu_{0}}|\vec{E} \times \vec{B}|
$$

But $\vec{B}=i k \vec{A} \sin \theta$ and $\vec{E}=i c \vec{A} \sin \theta$ so

$$
\frac{d P}{d \Omega}=\frac{c r^{2} k^{2}}{2 \mu_{0}}|\vec{A}|^{2} \sin ^{2} \theta=\frac{c \mu_{0} k^{2} d^{2} I_{0}^{2}}{32 \pi^{2}} \cos ^{2} \theta \sin ^{2} \theta=\frac{c \mu_{0} I_{0}^{2}}{8} \cos ^{2} \theta \sin ^{2} \theta
$$

c. Determine the total power radiated for the lowest multi-pole and the corresponding radiation resistance using both multi-pole moments from part a . compare with problem 9.16 b ; is there a paradox here?

$$
P=\int \frac{d P}{d \Omega} d \Omega=\frac{c \mu_{0} I_{0}^{2}}{8}(2 \pi) \int_{-1}^{1} \cos ^{2}(\theta) \sin ^{2}(\theta) d(\cos \theta)=\frac{c \mu_{0} \pi I_{0}^{2}}{15}
$$

Evaluate the integral as follows:

$$
\int_{-1}^{1} \cos ^{2}(\theta) \sin ^{2}(\theta) d(\cos \theta)=\int_{-1}^{1} \cos ^{2}(\theta)\left(1-\cos ^{2}(\theta)\right) d(\cos \theta)
$$

Let $\cos \theta=x$.

$$
\int_{-1}^{1}\left(x^{2}-x^{4}\right) d x=\frac{x^{3}}{3}-\left.\frac{x^{5}}{5}\right|_{x=-1} ^{1}=\frac{4}{15}
$$

In circuit analysis, we can write the power dissipated as

$$
P=R I_{0}^{2}
$$

Plug in the power radiated and solve for $R$.

$$
R=\frac{c \mu_{0} \pi}{15}=155 \Omega
$$

No paradox because interference of higher multi-poles is possible.

## Bonus Section: Broadcasting Westward

A professor posed once posed this question to me:
Suppose you had a city on the western shore of a large lake and that you are commissioned to design an antenna arrangement which would broadcast westward over the suburbs and waste as little power as possible by not broadcasting over the lake. Can this be done? How?
Obviously, by asking how, I have given you the answer to the first part. It's a bit difficult to understand the solution without diagrams so I'll put some diagrams here later. Position two antenna along the east-west axis and separate them by a distance $\frac{\lambda}{4}$. Now, delay the westward antenna by $\frac{\lambda}{4}$.
Here's what happens. The signal first appears at the eastern antenna. It propagates outward in all directions. When the pulse has traveled $\frac{\lambda}{4}$ westward, it passes the other antenna. At this moment, the second antenna emits the delayed signal. Both signals propagate in phase westwardly and so constructively interfere. Things are different on the eastward direction. By the time the second pulse reaches the first antenna the two signals are $\frac{\lambda}{2}$ out of phase and will destructively interfere. Thus, the eastward signal will be greatly diminished. According to the prof. who asked me this question, this is roughly the set up atop the Sears tower in Chicago.

## Problem 10.1

a. Show that for an arbitrary initial polarization, the scattering cross section of a perfectly conducting sphere of radius $a$, summed over outgoing polarizations, is given in the long-wavelength limit by

$$
\left(\frac{d \sigma}{d \Omega}\right)_{T o t}=k^{4} a^{6}\left[\frac{5}{4}-\left[\epsilon_{0} \cdot n\right]^{2}-\frac{1}{4}\left[n \cdot\left(n_{0} \times \epsilon_{0}\right)\right]^{2}-n_{0} \cdot n\right]
$$

where $n_{0}$ and $n$ are the directions of the incident and scattered radiations, respectively, while $\epsilon_{0}$ is the (perhaps complex) unit polarization vector of the incident radiation ( $\epsilon_{0}^{*} \cdot \epsilon_{0}=1 ; n \cdot \epsilon_{0}=0$ ).
O.K. This problem's a monster, a veritable Ungeheuer! The basic idea behind the problem is simple, and a college freshman with knowledge of high school algebra and a vague idea of how to manipulate vectors could quite conceivably solve this. Notwithstanding, the algebra is horrible, and algebra has been know to topple even the greatest physicists.
First, we will drop the vector notation. It should be obvious that all the $n$ 's and all the $\epsilon$ 's are unit vectors.
a. An unpolarized beam is scattered by a conducting sphere of radius $a$. From the text,

$$
\frac{d \sigma}{d \Omega}=k^{4} a^{6}\left[\hat{\epsilon}_{\text {out }}^{*} \cdot \hat{\epsilon}_{0}-\frac{1}{2}\left(n \times \hat{\epsilon}_{\text {out }}^{*}\right) \cdot\left(n_{0} \times \hat{\epsilon}_{0}\right)\right]^{2}
$$

It is a bit easier to work with dot products instead of cross products. Use the vector identity, $(\vec{A} \times \vec{B}) \cdot(\vec{C} \times \vec{D})=(\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D})-(\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C})$, to get

$$
\begin{equation*}
\frac{d \sigma}{d \Omega}=k^{4} a^{6}\left[\left(\hat{\epsilon}_{o u t}^{*} \cdot \hat{\epsilon}_{0}\right)\left[1-\frac{1}{2}\left(n \cdot n_{0}\right)\right]+\frac{1}{2}\left(n \cdot \hat{\epsilon}_{0}\right)\left(\hat{\epsilon}_{\text {out }}^{*} \cdot n_{0}\right)\right]^{2} \tag{5}
\end{equation*}
$$

Look at Jackson's diagram which I have included for convenience here. Notice that $n_{0} \cdot n=\cos \theta$.
First, construct an orthonormal basis. The most obvious unit vectors to use are one parallel to the incident wave vector,

$$
n_{0}
$$

one perpendicular to the scattering plane,

$$
\frac{n \times n_{0}}{\sin \theta}
$$



Figure 10.1 Polarization and propagation vectors for the incident and scattered radiation.

Figure 2: Jackson's insightful diagram.
and the third orthogonal to the first two,

$$
\frac{n-\left(n \cdot n_{0}\right) n_{0}}{\sin \theta}
$$

In case it is not obvious, the Graham-Schmidt process gave me the third vector. You can check for yourself to see that these vectors are orthogonal and normalized $(\hat{v} \cdot \hat{v}=1)$. The vector identity given earlier is useful for this. The most general incident scattering wave polarization can be written in terms of these three unit vectors.

$$
\epsilon_{0}=A\left(\frac{n \times n_{0}}{\sin \theta}\right)+B\left(n_{0}\right)+\Gamma\left(\frac{n-\left(n \cdot n_{0}\right) n_{0}}{\sin \theta}\right)
$$

And the most general scattered wave polarization vector can be expressed in terms of the same orthogonal basis.

$$
\epsilon_{\text {out }}^{*}=\epsilon_{\perp(1)}^{*}\left(\frac{n \times n_{0}}{\sin \theta}\right)+\epsilon_{\|(1)}^{*}\left(n_{0}\right)+\epsilon_{\|(2)}^{*}\left(\frac{n-\left(n \cdot n_{0}\right) n_{0}}{\sin \theta}\right)
$$

The parallel and perpendicular symbols refer to the polarizations orientation with respect to the scattering plane. We will use the following later:

$$
\hat{\epsilon}_{\perp}^{*}=\epsilon_{\perp(1)}^{*}\left(\frac{n \times n_{0}}{\sin \theta}\right)
$$

And

$$
\epsilon_{\|}^{*}=\epsilon_{\|(1)}^{*}\left(n_{0}\right)+\epsilon_{\|(2)}^{*}\left(\frac{n-\left(n \cdot n_{0}\right) n_{0}}{\sin \theta}\right)
$$

Proceed by determining the coefficients for the incident wave. We do this be doting the incident wave vector by our basis vectors. Remember that Jackson gives us $n_{0} \cdot \hat{\epsilon}_{0}=0$. We realize immediately that $B=0$. The other components are

$$
\Gamma=\hat{\epsilon}_{0} \cdot\left[\frac{n-\left(n \cdot n_{0}\right) n_{0}}{\sin \theta}\right]=\frac{1}{\sin \theta}\left[n \cdot \hat{\epsilon}_{0}-\left(n \cdot n_{0}\right)\left(n_{0} \cdot \hat{\epsilon}_{0}\right)\right]=\frac{1}{\sin \theta} n \cdot \hat{\epsilon}_{0}
$$

And

$$
A=\hat{\epsilon}_{0} \cdot \frac{n \times n_{0}}{\sin \theta}
$$

Calculate the scattering cross section for an arbitrarily polarized beam is done with the average of the incoming polarization and then the sum of the outgoing polarizations. That means that the total cross section is the sum of the cross sections for the two final polarization states. These states correspond to polarizations perpendicular and parallel to the scattering plane.

$$
\left(\frac{d \sigma}{d \Omega}\right)_{T o t}=\left(\frac{d \sigma}{d \Omega}\right)_{\|}+\left(\frac{d \sigma}{d \Omega}\right)_{\perp}
$$

In order to evaluate the cross sections, it will be helpful to know the following first: $\epsilon_{\|}^{*} \cdot \epsilon_{0}, \epsilon_{\perp}^{*} \cdot \epsilon_{0}, n \times \epsilon_{\|}^{*}, n \times \epsilon_{\perp}^{*}$, and $n_{0} \times \epsilon_{0}$. Rewrite the incident polarization by putting $\Gamma$ and $A$ in explicitly.

$$
\epsilon_{0}=\frac{1}{\sin ^{2} \theta}\left(n \cdot \epsilon_{0}\right)\left[n-\left(n_{0} \cdot n\right) n_{0}\right]+\frac{1}{\sin ^{2} \theta}\left[\left(n_{0} \times n\right) \cdot \epsilon_{0}\right]\left(n_{0} \times n\right)
$$

Now, take the relevant dot products.

$$
\left(n \cdot \hat{\epsilon}_{0}\right)=\frac{n \cdot \hat{\epsilon}_{0}}{\sin ^{2} \theta}\left[1-\left(n_{0} \cdot n\right)^{2}\right]=\frac{n \cdot \hat{\epsilon}_{0}}{\sin ^{2} \theta}\left(1-\cos ^{2} \theta\right)=n \cdot \hat{\epsilon}_{0}
$$

And

$$
\epsilon_{\|}^{*} \cdot \epsilon_{0}=\frac{1}{\sin ^{2} \theta}\left[-\left(n \cdot \hat{\epsilon}_{0}\right)\left(\hat{\epsilon}^{*} \cdot n_{0}\right)\left(n_{0} \cdot n\right)\right]=\frac{1}{\sin \theta}\left(n \cdot \hat{\epsilon}_{0}\right)\left(n_{0} \cdot n\right)
$$

And

$$
\epsilon_{\perp}^{*} \cdot \epsilon_{0}=\frac{1}{\sin ^{2} \theta}\left[\hat{\epsilon}_{\perp}^{*} \cdot\left(n_{0} \times n\right)\right]\left[\left(n_{0} \times n\right) \cdot \hat{\epsilon}_{0}\right]=\frac{1}{\sin \theta}\left[\left(n_{0} \times n\right) \cdot \hat{\epsilon}_{0}\right]
$$

We can also find

$$
\begin{gathered}
\epsilon_{\|}{ }^{*} \cdot n_{0}=-\sin \theta \\
\epsilon_{\perp}{ }^{*} \cdot\left(n_{0} \times n\right)=\sin \theta
\end{gathered}
$$

Now, we have all the dot products needed to find the cross sections.
For the parallel case, the scattering cross section is equation 5 with only $\hat{\epsilon}_{\|}$ in the final polarization.

$$
\begin{array}{r}
\left(\frac{d \sigma}{d \Omega}\right)_{\|}=k^{4} a^{6}\left[\left(\epsilon_{\|}^{*} \cdot \epsilon_{0}\right)\left[1-\frac{1}{2}\left(n \cdot n_{0}\right)\right]+\frac{1}{2}\left(n \cdot \epsilon_{0}\right)\left(\epsilon_{\|}^{*} \cdot n_{0}\right)\right]^{2} \\
=k^{4} a^{6}\left[\frac{1}{\sin \theta}\left(n \cdot \hat{\epsilon}_{0}\right)\left(n_{0} \cdot n\right)\left[1-\frac{1}{2}\left(n_{0} \cdot n\right)\right]+\frac{1}{2}\left(n \cdot \hat{\epsilon}_{0}\right)[-\sin \theta]\right]^{2} \\
=k^{4} a^{6}\left[\left(n \cdot \hat{\epsilon}_{0}\right) \frac{\cos \theta-\frac{1}{2}\left(\cos ^{2} \theta+\sin ^{2} \theta\right)}{\sin \theta}\right]^{2} \\
=k^{4} a^{6}\left[\left(n \cdot \hat{\epsilon}_{0}\right) \frac{\cos \theta-\frac{1}{2}}{\sin \theta}\right]^{2}
\end{array}
$$

For the perpendicular case, we do the same as above but instead of $\hat{\epsilon}_{\|}$, we have $\hat{\epsilon}_{\perp}$ in the final polarization.

$$
\begin{aligned}
\left(\frac{d \sigma}{d \Omega}\right)_{\perp}=k^{4} a^{6}\left[\left(\epsilon_{\perp}^{*} \cdot \epsilon_{0}\right)[ \right. & \left.\left.-\frac{1}{2}\left(n \cdot n_{0}\right)\right]+\frac{1}{2}\left(n \cdot \hat{\epsilon}_{0}\right)\left(\hat{\epsilon}_{\perp}^{*} \cdot n_{0}\right)\right]^{2} \\
& =k^{4} a^{6}\left[\left(\hat{\epsilon}_{\perp}^{*} \cdot \hat{\epsilon}_{0}\right)\left[1-\frac{1}{2}\left(n \cdot n_{0}\right)\right]\right]^{2} \\
=k^{4} a^{6}[ & \left.\frac{1}{\sin \theta}\left[\left(n_{0} \times n\right) \cdot \hat{\epsilon}_{0}\right]\left[1-\frac{1}{2} n_{0} \cdot n\right]\right]^{2} \\
& =k^{4} a^{6}\left[\left[\left(n_{0} \times n\right) \cdot \hat{\epsilon}_{0}\right] \frac{1-\frac{1}{2} \cos \theta}{\sin \theta}\right]^{2}
\end{aligned}
$$

We add these to get the total cross section.

$$
\left(\frac{d \sigma}{d \Omega}\right)_{T o t}=\frac{k^{4} a^{6}}{\sin ^{2} \theta}\left[\left[n \cdot \hat{\epsilon}_{0}\right]^{2}\left[\cos \theta-\frac{1}{2}\right]^{2}+\left[\left(n_{0} \times n\right) \cdot \hat{\epsilon}_{0}\right]^{2}\left[1-\frac{1}{2} \cos \theta\right]^{2}\right]
$$

Multiply out the squares.

$$
\begin{aligned}
& \left(\frac{d \sigma}{d \Omega}\right)_{T o t}=\frac{k^{4} a^{6}}{\sin ^{2} \theta}\left[\left[n \cdot \hat{\epsilon}_{0}\right]^{2}\left[\left(\cos ^{2} \theta-1\right)-\cos \theta+\frac{5}{4}\right]\right] \\
& \quad+\frac{k^{4} a^{6}}{\sin ^{2} \theta}\left[\left[\left(n_{0} \times n\right) \cdot \hat{\epsilon}_{0}\right]^{2}\left[\frac{1}{4}\left(\cos ^{2} \theta-1\right)+\frac{5}{4}-\cos \theta\right]\right]
\end{aligned}
$$

And then, with some algebra,

$$
\left(\frac{d \sigma}{d \Omega}\right)_{T o t}=k^{4} a^{6}\left[-\left[n \cdot \hat{\epsilon}_{0}\right]^{2}-\frac{1}{4}\left[\left(n_{0} \times n\right) \cdot \hat{\epsilon}_{0}\right]^{2}+\frac{\frac{5}{4}-\cos \theta}{1-\cos ^{2} \theta}\left(\left[n \cdot \hat{\epsilon}_{0}\right]^{2}+\left[\left(n_{0} \times n\right) \cdot \hat{\epsilon}_{0}\right]^{2}\right)\right]
$$

Recall that we were given

$$
\epsilon_{0}^{*} \cdot \epsilon_{0}=1 \rightarrow 1=\left[\epsilon_{0 \|}\right]^{2}+\left[\epsilon_{0 \perp}\right]^{2}
$$

This means that

$$
\frac{1}{\sin ^{2} \theta}\left[n \cdot \hat{\epsilon}_{0}\right]^{2}+\frac{1}{\sin ^{2} \theta}\left[\left(n_{0} \times n\right) \cdot \hat{\epsilon}_{0}\right]^{2}=1
$$

Finally, we can report the total cross section.

$$
\begin{equation*}
\left(\frac{d \sigma}{d \Omega}\right)_{T o t}=k^{4} a^{6}\left[\frac{5}{4}-\left[\epsilon_{0} \cdot n\right]^{2}-\frac{1}{4}\left[n \cdot\left(n_{0} \times \epsilon_{0}\right)\right]^{2}-n_{0} \cdot n\right] \tag{6}
\end{equation*}
$$

Where we replaced $\cos \theta$ with $n_{0} \cdot n$.
b. If the incident radiation is linearly polarized, show that the cross section is

$$
\left(\frac{d \sigma}{d \Omega}\right)=k^{4} a^{6}\left[\frac{5}{8}\left(1+\cos ^{2} \theta\right)-\cos \theta-\frac{3}{8} \sin ^{2} \theta \cos 2 \phi\right]
$$

where $n \cdot n_{0}=\cos \theta$ and the azimuthal angle $\phi$ is measured from the direction of the linear polarization.
It is a simple matter of geometry to determine the following dot and cross products. I'll give you a diagram someday, but for now, you've got to draw this one yourself.

$$
\begin{gathered}
\epsilon_{0} \cdot n=\sin \phi \sin \theta \\
n \cdot\left(n_{0} \times \epsilon_{0}\right)=\epsilon_{0} \cdot\left(n \times n_{0}\right)=\epsilon_{0} \cdot \hat{v} \sin \theta=\sin \theta \cos \phi
\end{gathered}
$$

Once we have these products, part b is simply a matter of trigonometric formulae and algebraic manipulations. Consider the term in brackets from equation 6 , and write the newly revealed angles in.

$$
\begin{aligned}
& {\left[\frac{5}{4}-\left[\epsilon_{0} \cdot n\right]^{2}-\frac{1}{4}\left[n \cdot\left(n_{0} \times \epsilon_{0}\right)\right]^{2}-n_{0} \cdot n\right]=} \\
& {\left[\frac{5}{4}-\sin ^{2} \phi \sin ^{2} \theta-\frac{1}{4} \sin ^{2} \theta \cos ^{2} \phi-\cos \theta\right]}
\end{aligned}
$$

I'm going to fly through this algebra. To start off, I will use $\cos 2 \alpha=$ $2 \cos ^{2} \alpha-1=1-2 \sin ^{2} \alpha$. It should be clear what's going on.

$$
\begin{array}{r}
{\left[\frac{5}{4}-\sin ^{2} \phi \sin ^{2} \theta-\frac{1}{4} \sin ^{2} \theta \cos ^{2} \phi-\cos \theta\right]} \\
=\left[\frac{5}{4}-\frac{1}{2}(1-\cos 2 \phi) \sin ^{2} \theta-\frac{1}{8} \sin ^{2} \theta(1-\cos 2 \phi)-\cos \theta\right] \\
=\left[\frac{5}{8}\left(1+\cos ^{2} \theta\right)-\cos \theta-\frac{3}{8} \sin ^{2} \theta \cos 2 \phi\right] \\
=\left[\frac{5}{4}-\frac{1}{2}(1+\cos 2 \phi) \sin ^{2} \theta-\frac{1}{8} \sin ^{2} \theta(1-\cos 2 \phi)-\cos \theta\right] \\
=\left[\frac{5}{4}-\frac{5}{8} \sin ^{2} \theta-\frac{3}{8} \sin ^{2} \theta \cos 2 \phi-\cos \theta\right] \\
=\left[\frac{5}{4}-\frac{5}{8}\left(1-\cos ^{2} \theta\right)-\frac{3}{8} \sin ^{2} \cos 2 \phi-\cos \theta\right] \\
=\left[\frac{5}{8}\left(1+\cos ^{2} \theta\right)-\cos \theta-\frac{3}{8} \sin ^{2} \theta \cos 2 \phi\right]
\end{array}
$$

Then, we have what Jackson wants.

$$
\left(\frac{d \sigma}{d \Omega}\right)=k^{4} a^{6}\left[\frac{5}{8}\left(1+\cos ^{2} \theta\right)-\cos \theta-\frac{3}{8} \sin ^{2} \theta \cos 2 \phi\right]
$$

## Problem 10.11

A perfectly conducting flat screen occupies half of the $x-y$ plane (i.e., $x<0$ ). A plane wave of intensity $I_{0}$ and wave number $k$ is incident along the $z$ axis from the region $z<0$. discuss the values of the diffracted fields in the plane parallel to the $x-y$ plane defined by $z=Z>0$. Let the coordinates of the observation point by $(X, 0, Z)$.

a. Show that, for the usual scalar Kirchoff approximation and in the limit $Z \gg X$ and $\sqrt{k Z} \gg 1$, the diffracted field is

$$
\Psi=\sqrt{I_{0}}\left(\frac{1+i}{2 i}\right) e^{i k Z-i \omega t} \sqrt{\frac{2}{\pi}} \int_{-\Xi}^{\infty} e^{i u^{2}} d u
$$

where $\Xi=X\left(\frac{k}{2 Z}\right)^{\frac{1}{2}}$.

$$
\Psi\left(r_{0}\right)=\frac{k}{2 \pi i} \sqrt{I_{0}} \int_{\text {Aperture }} \frac{e^{i k r_{p}}}{r_{p}} d A^{\prime}
$$

$r_{0}$ is the observation point, and $r_{p}=\sqrt{\left(x^{\prime}-X\right)^{2}+\left(y^{\prime}-Y\right)^{2}+\left(z^{\prime}-Z\right)^{2}}$ is the distance from the area point at the aperture to the observation point. The small letters denote the aperture values while the large letters denote values at the observation point. $d A^{\prime}=d x^{\prime} d y^{\prime}$ in this case because the screen is in the $x y$ plane.
I proceed first by evaluating the integral over the $y$ coordinate.

$$
I_{1}=\int_{\infty}^{\infty} \frac{e^{i k r_{p}}}{r_{p}} d y^{\prime}
$$

I exploit the symmetry of the integral about $y=0$, and replace $\rho^{2}=\left(x^{\prime}-\right.$ $X)^{2}+\left(z^{\prime}-Z\right)^{2}$.

$$
I_{1}=2 \int_{0}^{\infty} \frac{e^{i k \sqrt{\left(y^{\prime}-Y\right)^{2}+\rho^{2}}}}{\sqrt{\left(y^{\prime}-Y\right)^{2}+\rho^{2}}} d y^{\prime}
$$

Substitute $\nu=\sqrt{\left(y^{\prime}-Y\right)^{2}+\rho^{2}}$.

$$
I_{1}=2 \int_{\rho}^{\infty} \frac{e^{i k \nu}}{\sqrt{\nu^{2}-\rho^{2}}} d \nu
$$

Remember from basic calculus,

$$
\begin{aligned}
\int_{0}^{\infty} \frac{\sin (A x)}{\sqrt{x^{2}-1}} d x & =\frac{\pi}{2} J_{0}(A) \\
\int_{0}^{\infty} \frac{\cos (A x)}{\sqrt{x^{2}-1}} d x & =-\frac{\pi}{2} N_{0}(A)
\end{aligned}
$$

$J_{0}$ is a Bessel function and $N_{0}$ is a Neumann function. I will use these to reduce the integral to a more tractable form. By Euler's handy formula, $e^{i x}=\cos x+i \sin x$. so we can write

$$
I_{1}=2 \int_{\rho}^{\infty} \frac{e^{i k \nu}}{\sqrt{\nu^{2}-\rho^{2}}} d \nu=2 \int_{\rho}^{\infty} \frac{[\cos (k \nu)+i \sin (k \nu)]}{\sqrt{\nu^{2}-\rho^{2}}} d \nu
$$

Let $\xi=\nu / \rho$ and $d \xi=\frac{1}{\rho} d \nu$.

$$
I_{1}=2 \int_{1}^{\infty} \frac{[\cos (k \rho \xi)+i \sin (k \rho \xi)]}{\sqrt{\xi^{2}-1}} d \xi=2\left[-\frac{\pi}{2} N_{0}(k \rho)+i \frac{\pi}{2} J_{0}(k \rho)\right]
$$

And so the first part of the surface integral is done.
Now, I will attempt to integrate over $d x^{\prime}$. Don't forget $\rho$ is a function of $x^{\prime}$.

$$
I_{2}=\int_{0}^{\infty}-\pi N_{0}(k \rho)+i \pi J_{0}(k \rho) d x^{\prime}=i \pi \int J_{0}(k \rho)+i N_{0}(k \rho) d x^{\prime}
$$

In the limit $\sqrt{k Z} \gg 1 \rightarrow k Z \gg 1$ and $\rho k \gg 1$, the Bessel function and its friend can be approximated by the following:

$$
J_{0}(A) \simeq \sqrt{\frac{2}{\pi A}} \cos \left(A-\frac{\pi}{4}\right)
$$

$$
N_{0}(A) \simeq-\sqrt{\frac{2}{\pi A}} \sin \left(A-\frac{\pi}{4}\right)
$$

And the integral reduces to

$$
I_{2}=\int_{0}^{\infty} i \pi \sqrt{\frac{2}{\pi k \rho}}\left[\cos \left(k \rho-\frac{\pi}{4}\right)+i \sin \left(k \rho-\frac{\pi}{4}\right)\right] d x^{\prime}
$$

Which easily reduces to

$$
I_{2}=\int_{0}^{\infty} i \sqrt{\frac{2 \pi}{k \rho}} e^{i\left(\rho k-\frac{\pi}{4}\right)} d x^{\prime}=i \sqrt{2 \pi} \int_{0}^{\infty} e^{i\left(\rho k-\frac{\pi}{4}\right)} \sqrt{\frac{1}{k \rho}} d x^{\prime}
$$

Lest I loose track of all the coefficients, I'll rewrite $\Psi$.

$$
\Psi=\frac{k}{2 \pi i} \sqrt{I_{0}} i \sqrt{2 \pi} \int_{0}^{\infty} \frac{e^{i\left(\rho k-\frac{\pi}{4}\right)}}{\sqrt{\rho k}} d x^{\prime}=k \sqrt{\frac{I_{0}}{2 \pi}} e^{-i \frac{\pi}{4}} \int_{0}^{\infty} \frac{e^{\left[i k \sqrt{\left.\left(x^{\prime}-X\right)^{2}+\left(z^{\prime}-Z\right)^{2}\right]}\right.}}{\sqrt{k \sqrt{\left(x^{\prime}-X\right)^{2}+\left(z^{\prime}-Z\right)^{2}}}} d x^{\prime}
$$

I have written $\rho$ in explicitly to remind us that $\rho$ depends on $x^{\prime}$. Now, I label the integral as $I_{3}$ and tackle this integration.

$$
I_{3}=\int_{0}^{\infty} \frac{e^{\left[i k \sqrt{\left(x^{\prime}-X\right)^{2}+\left(z^{\prime}-Z\right)^{2}}\right]}}{\sqrt{k \sqrt{\left(x^{\prime}-X\right)^{2}+\left(z^{\prime}-Z\right)^{2}}}} d x^{\prime}
$$

So far, I haven't make use of the fact that $z^{\prime}=0$. I'll do that now. If $\left(x^{\prime}-X\right) \ll Z$, we can expand $\sqrt{\left(x^{\prime}-X\right)^{2}+Z^{2}} \simeq Z+\frac{\left(x^{\prime}-X\right)^{2}}{2 Z}$. So

$$
I_{3}=\frac{e^{i k Z}}{\sqrt{k Z}} \int_{-\Xi}^{\infty} \sqrt{\frac{2 Z}{k}} e^{i u^{2}} d u
$$

where $u=\sqrt{\frac{k}{2 Z}}\left(x^{\prime}-X\right)$, and the limits of integration have been changed accordingly, $\Xi=X \sqrt{k /(2 Z)}$. This gives the result:

$$
\Psi=k \sqrt{\frac{I_{0}}{2 \pi}} e^{-i \frac{\pi}{4}} \frac{e^{i k Z}}{\sqrt{k Z}} \sqrt{\frac{2 Z}{k}} \int_{-\Xi}^{\infty} e^{i u^{2}} d u
$$

A little work with an Argand diagram should convince you that

$$
e^{-i \frac{\pi}{4}}=\frac{1}{\sqrt{2}}-i \frac{1}{\sqrt{2}}=\sqrt{2}\left(\frac{1+i}{2 i}\right)
$$

and then, $\Psi$ reduces to Jackson's result.

$$
\Psi=\sqrt{I_{0}}\left(\frac{1+i}{2 i}\right) e^{i k Z-i \omega t} \sqrt{\frac{2}{\pi}} \int_{-\Xi}^{\infty} e^{i u^{2}} d u
$$

where $\Xi=X\left(\frac{k}{2 Z}\right)^{\frac{1}{2}}$. Note: I didn't assume time dependence from the start, but if I did the derivation would be the same. I would have simply factored the $e^{-i \omega t}$ out from the start. So I just put it back here.
b. Find the intensity. Determine the asymptotic behavior of $I$ for $\xi$ large and positive (illuminated region) and $\xi$ large and negative (shadow region). what is the value of $I$ at $X=0$ ? Make a sketch of $I$ as a function of $X$ for fixed $Z$.
We need to rewrite $I_{4}$ is a suggestive way.

$$
I_{4}=\int_{-\Xi}^{\infty} e^{i u^{2}} d u
$$

Everybody should know the friendly Fresnel Integrals:

$$
\begin{aligned}
& C(\lambda)=\int_{0}^{\lambda} \cos \left(\frac{\pi x^{2}}{2}\right) d x \\
& S(\lambda)=\int_{0}^{\lambda} \sin \left(\frac{\pi x^{2}}{2}\right) d x
\end{aligned}
$$

And using Euler's handy relationship,

$$
\int_{0}^{\lambda} e^{i \pi \frac{x^{2}}{2}} d x=C(\lambda)+i S(\lambda)
$$

In our case.

$$
\int_{-\Xi}^{\infty} e^{i u^{2}} d u=\sqrt{\frac{\pi}{2}}[C(\infty)+i S(\infty)-C(-\Xi)-i S(-\Xi)]
$$

I will use the symmetry of $C(x)$ and $S(x)$, namely, $C(x)=-C(-x)$ and $S(x)=-S(-x)$ to get rid of all the unwanted minus signs.

$$
\int_{-\Xi}^{\infty} e^{i u^{2}} d u=\sqrt{\frac{\pi}{2}}[C(\infty)+i S(\infty)+C(\Xi)+i S(\Xi)]
$$

To find $C(x)$ and $S(x)$ at infinity, we need $\lim _{t \rightarrow \pm \infty} C(t)= \pm \frac{1}{2}$ and $\lim _{t \rightarrow \pm \infty} S(t)=$ $\pm \frac{1}{2} . I_{4}$ is evidently representable by $\frac{1}{2}(1+i)+C(\Xi)+i S(\Xi)$. The intensity is given by $|\Psi|^{2}$ so

$$
\begin{array}{r}
\mathcal{I}=\left\lvert\, \sqrt{\frac{2}{\pi}} \sqrt{\left.I_{0}\left(\frac{1+i}{2 i}\right) e^{i k Z-i \omega t}\right|^{2}\left[\frac{1+i}{2}+C(\Xi)+i S(\Xi)\right]^{2}=}\right. \\
\left(\frac{2}{\pi} I_{0}\right) \frac{\pi}{2}\left[\left(C(\Xi)+\frac{1}{2}\right)^{2}+\left(S(\Xi)+\frac{1}{2}\right)^{2}\right]
\end{array}
$$

And finally, we have what Jackson wants.

$$
\mathcal{I}=\frac{I_{0}}{2}\left[\left(C(\Xi)+\frac{1}{2}\right)^{2}+\left(S(\Xi)+\frac{1}{2}\right)^{2}\right]
$$

As $\Xi \rightarrow \infty+, \mathcal{I} \rightarrow I_{0}$, and we have a bright spot. As $\Xi \rightarrow \infty-, \mathcal{I} \rightarrow 0$, and we have a shadow. At $X=0, \Xi=0$, and $\mathcal{I}=\frac{I_{0}}{4}$.
The graph is coming soon!


## Bonus Section: A Review of Lorentz Invariant Quantities

Relativistic notation is a mess, and admittedly Jackson doesn't do that poorly trying to straighten things out. In order to keep track of the pesky minus sign in the Minkowski metric, $\sigma^{2}=t^{2}-x^{2}-y^{2}-z^{2}$, we define two types of four tensor, covariant and contra-variant ${ }^{4}$. Contra-variant tensors are much like a regular old Euclidean tensor. The entries all have positive signs and transform as you'd expect (I'll get to that later). I keep the name and position of the indices straight by remembering how contradictory relativity first seemed, but since these vectors are easier to work with, I'll give them one thumb up and place their indices up. It's no surprise that the simpler named covariant vectors have tricky minus signs before the space coordinates. I'll put this covariant indices low because of this covert behavior. O.K. Enough silly semantics.

My purpose here is to review a bit of notation and to stress the usefulness of Lorentz invariants. First, accept $\frac{\partial x^{\alpha}}{\partial x^{\beta}}=\delta_{\alpha \beta}$. I think Goldstein discusses this in his sections on field theory, so I won't explain where this came from. Clearly, this is reasonable. A derivative of a constant is zero, and a derivative of a function by itself is one.
For a first rank tensor, the transformation rules are as follows: Covariant first ranked tensor,

$$
A^{\prime \alpha}=\frac{\partial x^{\prime \alpha}}{\partial x^{\beta}} A^{\beta}
$$

Contra-variant first ranked tensor,

$$
B_{\gamma}^{\prime}=\frac{\partial x^{\epsilon}}{\partial x^{\prime \gamma}} B_{\epsilon}
$$

And the scalar product, $B_{\alpha}^{\prime} A^{\alpha \prime}$,

$$
B_{\alpha}^{\prime} A^{\prime \alpha}=\frac{\partial x^{\epsilon}}{\partial x^{\prime \alpha}} B_{\epsilon} \frac{\partial x^{\prime \alpha}}{\partial x^{\beta}} A^{\beta}=\frac{\partial x^{\epsilon}}{\partial x^{\beta}} B_{\epsilon} A^{\beta}=\delta_{\epsilon \beta} B_{\epsilon} A^{\beta}=B_{\beta} A^{\beta}
$$

is invariant under Lorentz transformations. For example, the mass of a particle is a Lorentz invariant. $\wp_{\mu}^{\prime} \wp^{\mu \prime}=\wp_{\mu} \wp^{\mu}=m^{2}$.
For second rank tensors, we can devise similar rules. First, for the completely covariant object

$$
C^{\prime \alpha \beta}=\frac{\partial x^{\prime \alpha}}{\partial x^{\gamma}} \frac{\partial x^{\prime \beta}}{\partial x^{\epsilon}} C^{\prime \gamma \epsilon}
$$

[^3]And now for the completely contra-variant monster

$$
D_{\alpha \beta}^{\prime}=\frac{\partial x^{\zeta}}{\partial x^{\prime \alpha}} \frac{\partial x^{\eta}}{\partial x^{\prime \beta}} D_{\zeta \eta}^{\prime}
$$

And I suppose we could consider a mixed object, but I have grown tired of writing all these indices. The scalar product of two second rank tensors is invariant under Lorentz transformations:

$$
\begin{gathered}
D_{\alpha \beta}^{\prime} C^{\prime \alpha \beta}=\frac{\partial x^{\zeta}}{\partial x^{\prime \alpha}} \frac{\partial x^{\eta}}{\partial x^{\prime \beta}} D_{\zeta \eta}^{\prime} \frac{\partial x^{\prime \alpha}}{\partial x^{\gamma}} \frac{\partial x^{\prime \beta}}{\partial x^{\epsilon}} C^{\prime \gamma \epsilon}= \\
\frac{\partial x^{\zeta}}{\partial x^{\gamma}} \frac{\partial x^{\eta}}{\partial x^{\epsilon}} D_{\zeta \eta} C^{\gamma \epsilon}=\delta_{\zeta \gamma} \delta_{\eta \epsilon} D_{\zeta \eta} C^{\gamma \epsilon}=D_{\zeta \eta} C^{\zeta \eta}
\end{gathered}
$$

For example, magnetic and electric dipoles can be expressed by a tensors, $M_{\mu \nu}$, and $F^{\alpha \beta}$ in the electro-magnetic field tensor. $U_{\text {interaction }}=\frac{1}{2} M_{\mu \nu} F^{\mu \nu}=$ $\frac{1}{2} M_{\mu \nu}^{\prime} F^{\prime \mu \nu}$; id est the interaction energy is invariant under Lorentz transformations.

## Problem 11.5

A coordinate system $K^{\prime}$ moves with a velocity $\vec{v}$ relative to another system $K$. In $K^{\prime}$, a particle have velocity $\vec{u}^{\prime}$ and an acceleration $\vec{a}^{\prime}$. Find the Lorentz transformation law for accelerations, and show that in the system $K$ the components of acceleration parallel and perpendicular to $\vec{v}$ are as I will derive.
To make life easier on me, I'll omit the vector symbols on the vectors in this problem. They should be obvious anyway.
To solve this problem, I must use the relationship

$$
\frac{d t^{\prime}}{d t}=\left(1-\frac{v^{2}}{c^{2}}\right)^{\frac{1}{2}}\left(1+\frac{v \cdot u^{\prime}}{c^{2}}\right)^{-1}
$$

So first, I'll derive this. From Jackson page 531, we have $u^{\prime}=c \frac{d x^{\prime}}{d t^{\prime}}$ which can be rearranged to give $d x^{\prime}=\frac{u^{\prime}}{c} d t^{\prime}$. Dot multiply both sides of this equation for $d x^{\prime}$ by $\beta=\frac{v}{c}$. Then, we have $\beta \cdot d x^{\prime}=\beta \cdot \frac{u^{\prime}}{c} d t^{\prime}=\frac{v \cdot u^{\prime}}{c^{2}} d t^{\prime}$. According to Jackson in section 11.4, we have the relationship $d t=\gamma\left(d t^{\prime}+\beta \cdot d x^{\prime}\right)$. As usual, $\gamma=\left(1-\frac{v^{2}}{c^{2}}\right)^{-\frac{1}{2}}$. Plug the equation for $\beta \cdot d x^{\prime}$ into the equation for $d t$, and then we get

$$
d t=\gamma\left(1+\frac{v \cdot u^{\prime}}{c^{2}}\right) d t^{\prime} \rightarrow \frac{d t^{\prime}}{d t}=\left(1-\frac{v^{2}}{c^{2}}\right)^{\frac{1}{2}}\left(1+\frac{v \cdot u^{\prime}}{c^{2}}\right)^{-1}
$$

As I intended to prove.
From Jackson 11.31, we have the velocity addition equation for parallel components of velocity

$$
u_{\|}=\frac{u_{\|}^{\prime}+v}{1+\frac{v \cdot u^{\prime}}{c^{2}}}
$$

Take the derivative with respect to $d t$.

$$
a_{\|}=\frac{d u_{\|}}{d t}=\frac{a_{\|}^{\prime}}{1+\frac{v \cdot u^{\prime}}{c^{2}}} \frac{d t^{\prime}}{d t}-\left(u_{\|}^{\prime}+v\right)\left(1+\frac{v \cdot u_{\|}^{\prime}}{c^{2}}\right)^{-2}\left(\frac{v}{c^{2}}\right) a_{\|}^{\prime} \frac{d t^{\prime}}{d t}
$$

With some rearrangement,

$$
\begin{aligned}
a_{\|}=\frac{d u_{\|}}{d t} & =\left(1+\frac{v \cdot u^{\prime}}{c^{2}}\right)\left(1+\frac{v \cdot u^{\prime}}{c^{2}}\right)^{-2} a_{\|}^{\prime} \frac{d t^{\prime}}{d t} \\
& -\left(\frac{u^{\prime} \cdot v}{c^{2}}+\frac{v^{2}}{c^{2}}\right)\left(1+\frac{v \cdot u_{\|}^{\prime}}{c^{2}}\right)^{-2} a_{\|}^{\prime} \frac{d t^{\prime}}{d t}
\end{aligned}
$$

And the inclusion of $\frac{d t^{\prime}}{d t}$, we have

$$
a_{\|}=\frac{1-\frac{v^{2}}{c^{2}}}{\left(1+\frac{v \cdot u_{\|}^{\prime}}{c^{2}}\right)^{2}} a_{\|}^{\prime} \frac{d t^{\prime}}{d t}=\frac{\left(1-\frac{v^{2}}{c^{2}}\right)^{\frac{1}{2}}}{\left(1+\frac{v \cdot u^{\prime}}{c^{2}}\right)^{3}} a_{\|}^{\prime}
$$

In 11.31, Jackson also reported that for the perpendicular components of velocity, the addition law is

$$
u_{\perp}=\frac{u_{\perp}^{\prime}}{\gamma\left(1+\frac{v \cdot u^{\prime}}{c^{2}}\right)}
$$

Once again, we take the derivative with respect to $d t$.

$$
a_{\perp}=\frac{d u_{\perp}}{d t}=\frac{a_{\perp}^{\prime}}{\gamma\left(1+\frac{v \cdot u^{\prime}}{c^{2}}\right)} \frac{d t^{\prime}}{d t}-\frac{u_{\perp}^{\prime}}{\gamma\left(1+\frac{v \cdot u^{\prime}}{c^{2}}\right)^{2}}\left(\frac{v \cdot a^{\prime}}{c^{2}}\right) \frac{d t^{\prime}}{d t}
$$

After we plug in $\frac{d t^{\prime}}{d t}$ explicitly,

$$
a_{\perp}=\frac{1-\frac{v^{2}}{c^{2}}}{\left(1+\frac{u \cdot v^{\prime}}{c^{2}}\right)^{3}}\left[a_{\perp}^{\prime}\left(1+\frac{v \cdot u^{\prime}}{c^{2}}\right)-\frac{u_{\perp}^{\prime}}{c^{2}}\left(v \cdot a^{\prime}\right)\right]
$$

Using the vector identity, $\vec{A} \times(\vec{B} \times \vec{C})=(A \cdot C) \vec{B}-(A \cdot B) \vec{C}$, we can write $v \times\left(a^{\prime} \times u^{\prime}\right)=\left(v \cdot u^{\prime}\right) a_{\perp}^{\prime}-\left(v \cdot a^{\prime}\right) u_{\perp}^{\prime}+$ canceling $a_{\|}^{\prime}$ and $u_{\|}^{\prime}$ components. Possibly, this might not be so obvious to you. Well, $v$ cross anything must be perpendicular to $v$. Therefore, the only vector components on the right side of the triple product must be perpendicular to $v$ or be pair ed in such a way as to cancel. So finally, I can report.

$$
a_{\perp}=\frac{1-\frac{v^{2}}{c^{2}}}{\left(1+\frac{u \cdot v^{\prime}}{c^{2}}\right)^{3}}\left[a_{\perp}^{\prime}+\frac{1}{c^{2}} v \times\left(a^{\prime} \times u^{\prime}\right)\right]
$$

And I have given Jackson what he wants.

## Problem 11.6

One set of twins is born in the year 2080. In the year 2100, NASA decides to do an experiment. The government furtively seizes one twin, throws him aboard a rocket bound for a distant star, and sends the rocket into space. The rocket accelerates at the acceleration of gravity, $g$, in its own rest frame. Although the twin is lonely, he won't be too uncomfortable. The ship accelerates in a straight-line path for 5 years (by its own clocks), decelerates at the same rate for five years, turns around, accelerates for 5 years, decelerates for 5 years, and lands on earth. The twin in the rocket is then 40 years old.
According to the twin on the rocket, the trip to the distant star and back lasts 20 years.

$$
a\left(t^{\prime}\right)=\left\{\begin{array}{l}
g, t^{\prime}<5 \\
-g, 5<t^{\prime}<15 \\
g, 15<t^{\prime}<20
\end{array}\right.
$$

a. What year is it on earth?

How much time will pass on the earth during this trip? Will the space bound twin ever see his brother again?
Consider the first leg. $a=g$ while $t^{\prime}=0$ to 5 . Let $t^{\prime}$ denote the time on the rocket and $t$ denote the time on the earth. There is a simple relationship from the Lorentz transformation equations between these times $t^{\prime}=\frac{t}{\gamma\left(t^{\prime}\right)}$. For infinitesimal intervals, we have $d t^{\prime}=\frac{d t}{\gamma\left(t^{\prime}\right)}$. The total time elapsed on the rocket is:

$$
\begin{gathered}
T_{\text {rocket }}=\int_{0}^{5} d t^{\prime}=5 \\
T_{\text {earth }}=\int_{0}^{?} d t=\int_{0}^{5} \gamma\left(t^{\prime}\right) d t^{\prime}
\end{gathered}
$$

To get $\gamma\left(t^{\prime}\right)$, we need to sum over possible velocities so I'll use rapidity ${ }^{5}$ which is easier to work with.
First, we need $d t^{\prime}$ in terms of rapidity. $\beta=\frac{v}{c} \rightarrow d \beta=\frac{d v^{\prime}}{d t^{\prime}} \frac{d t^{\prime}}{c}=\frac{g}{c} d t^{\prime}$. This gives us $d t^{\prime}=\frac{c}{g} d \beta$. Now, I need to figure out what $\gamma(\theta)$ is. Jackson, in one of his rare instructive moments, taught us that $\beta=\tanh (\theta)$. I can exploit

[^4]the additive properties of rapidity, $\theta$, to write
$$
\beta=\tanh \left(\sum \theta\right) \rightarrow \Delta \beta=\tanh (\Delta \theta)
$$

And so in the infinitesimal limit, I have $d \beta=\tanh (d \theta)$. I can compare this to the first equation for $d \beta=\frac{g}{c} d t^{\prime}$ to get an expression for $d t^{\prime}$.

$$
d t^{\prime}=\frac{c}{g} \tanh (d \theta)
$$

Now, I expand $\tanh (d \theta)=d \theta-\frac{d \theta^{3}}{3}$ and keep only the first term.

$$
d t^{\prime}=\frac{c}{g} d \theta
$$

Finally,

$$
\beta\left(t^{\prime}\right)=\tanh \left(\int d \theta\right)=\tanh \left(\int_{0}^{t^{\prime}} \frac{g}{c} d t^{\prime \prime}\right)=\tanh \left(\frac{g}{c} t^{\prime}\right)
$$

By the relationships given on Jackson 11.20, $\gamma\left(t^{\prime}\right)=\cosh \left(\frac{g}{c} t^{\prime}\right)$ and that $\beta\left(t^{\prime}\right) \gamma\left(t^{\prime}\right)=\sinh \left(\frac{g}{c} t^{\prime}\right)$. Putting all this together,

$$
\begin{aligned}
T_{\text {earth }} & =\int \gamma\left(t^{\prime}\right) d t^{\prime}=\int_{0}^{5} \cosh \left(\frac{g}{c} t^{\prime}\right) d t^{\prime} \\
& =\left.\frac{c}{g} \sinh \left(\frac{g}{c} t^{\prime}\right)\right|_{0} ^{5}=\frac{c}{g} \sinh \left(\frac{5 g}{c}\right)
\end{aligned}
$$

Take $g=10 \mathrm{~m} / \mathrm{sec}^{2}$ and $c=3 \times 10^{8} \mathrm{~m} / \mathrm{sec}$. Don't forget that 5 is in years, and the equation is wrong unless I covert 1 year $=1.5768 \times 10^{8}$ seconds. Then, $T_{\text {earth }}=91$ years. By symmetry, the next three legs must each take just as long. So the total time elapsed on the earth while the rocket twin makes a round trip is $4 \times 91=364$ years. The twin will come back home in the year $2100+365=2464$, and his brother will be dead.

## b. How far away from the earth did the ship travel?

I will use a similar technique.

$$
\begin{gathered}
x_{\text {earth }}=\int c \beta\left(t^{\prime}\right) d t=\int_{0}^{10} c \beta\left(t^{\prime}\right) \gamma\left({ }^{\prime} t\right) d t^{\prime} \\
x_{\text {earth }}=\int_{0}^{5} c \sinh \left(\frac{g}{c} t^{\prime}\right) d t^{\prime}-\int_{0}^{5} c \sinh \left(\frac{-g}{c} t^{\prime}\right) d t^{\prime} \\
\\
=\left.\frac{2 c^{2}}{g} \cosh \left(\frac{g}{c} t\right)\right|_{0} ^{5} \approx 168
\end{gathered}
$$

So the distance to the turning point is about 168 light years. This makes sense because if we assume that the ship was traveling at pretty much the speed of light for $2 \times 91$ years, the ship would have gone 182 light years. But obviously, the ship was going a little bit slower.

## Problem 11.7

In the reference frame $K$ two evenly matched sprinters are lined up a distance $d$ apart on the $y$ axis for a race parallel to the $x$ axis. Two starters, one beside each man (or woman), will fire their starting pistols at slightly different times, giving a handicap to the better of the two runners. The time difference in $K$ is $T$.
a. For what range of time differences will there be a reference frame $K^{\prime}$ in which there is no handicap, and for what range of time differences in there a frame $K^{\prime}$ in which there is a true (not apparent) handicap?
The question is about the non-synchronicity of events as witnessed in different Lorentz frame.
For simplicity and for symmetry reasons, we'll consider a Lorentz frame moving along the $y$-axis parallel to the starting line or perpendicular to the race path, $x$-axis.
In $S$, the race frame: $y_{A}=\frac{1}{2} d$ and $y_{B}=-\frac{1}{2} d$. The starting time for the first runner is $t_{B}=0$, and for the second runner is $t_{A}=T$.
In $S^{\prime}$, the arbitrary Lorentz frame moving with velocity $u$ along the $y$-axis relative to the race frame:

$$
T^{\prime}=\frac{T c^{2}-u d}{c^{2} \sqrt{1-\frac{u^{2}}{c^{2}}}}
$$

If the handicap is not real, we can find a frame in which $T^{\prime}=0$. This will be the case when $\Delta t=\frac{u d}{c^{2}}$. Since we have a range of possible frame speeds from 0 to $c$, we can find a corresponding range of possible delays, $T=0$ to $\frac{d}{c}$. For larger time delays, the runner is given a true handicap.
b. Determine explicitly the Lorentz transformation to the frame $K^{\prime}$ appropriate for each of the two possibilities in part a, finding the velocity of $K^{\prime}$ relative to $K$ and the space-time positions of each sprinter in $K^{\prime}$.
To find the Lorentz frame in which the time delay vanishes, we use the condition on $T$ and solve for $u$.

$$
u=\frac{c^{2} T}{d}
$$

Obtaining the Lorentz transformations is straightforward. We can write these
as matrices.

$$
\binom{t_{A}^{\prime}}{y_{A}^{\prime}}=\left(\begin{array}{cc}
\gamma & -\gamma \beta \\
-\gamma \beta & \gamma
\end{array}\right)\binom{t_{A}}{y_{A}}=\left(\begin{array}{cc}
\gamma & -\gamma \frac{c T}{d} \\
-\gamma \frac{c T}{d} & \gamma
\end{array}\right)\binom{T}{\frac{1}{2} d}
$$

And

$$
\binom{t_{B}^{\prime}}{y_{B}^{\prime}}=\left(\begin{array}{cc}
\gamma & -\gamma \beta \\
-\gamma \beta & \gamma
\end{array}\right)\binom{t_{B}}{y_{B}}=\left(\begin{array}{cc}
\gamma & -\gamma \frac{c T}{d} \\
-\gamma \frac{c T}{d} & \gamma
\end{array}\right)\binom{0}{-\frac{1}{2} d}
$$

With $\gamma=\frac{1}{\sqrt{1-\frac{c^{2} T^{2}}{d^{2}}}}$.

$$
\begin{aligned}
& y_{A}^{\prime}=\frac{\frac{1}{2} d-\frac{c^{2} T^{2}}{d}}{\sqrt{1-\frac{c^{2} T^{2}}{d^{2}}}} \\
& t_{A}^{\prime}=\frac{\frac{1}{2} T}{\sqrt{1-\frac{c^{2} T^{2}}{d^{2}}}} \\
& y_{B}^{\prime}=\frac{-\frac{1}{2} d}{\sqrt{1-\frac{c^{2} T^{2}}{d^{2}}}} \\
& t_{B}^{\prime}=\frac{\frac{1}{2} T}{\sqrt{1-\frac{c^{2} T^{2}}{d^{2}}}}
\end{aligned}
$$

To find the transformations in the true handicap case is also straightforward. I define $T=\frac{d}{c}+\epsilon$, where $\epsilon$ is the part of the handicap that will never be transformed entirely away.

$$
\begin{gathered}
t_{A}^{\prime}=\frac{\frac{d}{c}+\epsilon-\frac{1}{2} \frac{u d}{c^{2}}}{\sqrt{1-\frac{u^{2}}{c^{2}}}} \\
y_{A}^{\prime}=\frac{\frac{1}{2} d-u\left(\frac{d}{c}+\epsilon\right)}{\sqrt{1-\frac{u^{2}}{c^{2}}}} \\
t_{B}^{\prime}=\frac{\frac{1 u d}{2} \frac{d}{c^{2}}}{\sqrt{1-\frac{u^{2}}{c^{2}}}} \\
y_{B}^{\prime}=\frac{-\frac{1}{2} d}{\sqrt{1-\frac{u^{2}}{c^{2}}}}
\end{gathered}
$$

## Problem 11.13

An infinitely long straight wire of negligible cross-sectional area is at rest and has a uniform linear charge density $q_{0}$ in the inertial frame $K^{\prime}$. The frame $K^{\prime}$ (and the wire) move with velocity $\vec{v}$ parallel to the direction of the wire with respect to the laboratory frame $K$.

In $K$, we have a wire with charge density, $\lambda$, but no current density. Because of the obvious cylindrical symmetry, we'll use cylindrical coordinates. In $K^{\prime}$, we have a nonzero current density, $\vec{J}^{\prime} \neq 0$, and a charge density, $\lambda^{\prime}$. The velocity of frame $K^{\prime}$ with respect to $K$ is $\vec{v}=v \hat{z}$. Watch out because in this problem I start off using Jackson units, but then switch rather abruptly to S.I. units.
a. Write down the electric and magnetic fields in cylindrical coordinates in the rest frame of the wire. Using the Lorentz transformation properties of the fields, find the components of the electric and magnetic fields in the laboratory.
In $K$,

$$
E_{r}=\frac{q_{0}}{2 \pi \epsilon_{0} r}
$$

$E_{\theta}=0$ and $E_{z}=0$ by Gauss's law, and $\vec{B}=0$ from the fact that there is no current (real or displacement) in this frame. We can use the Lorentz transformations for the fields to get from one frame to another. They are:

$$
\begin{aligned}
& E^{\prime}=\gamma(E+\beta \times B)-\frac{\gamma^{2}}{\gamma+1} \beta(\beta \cdot E) \\
& B^{\prime}=\gamma(B-\beta \times E)-\frac{\gamma^{2}}{\gamma+1} \beta(\beta \cdot B)
\end{aligned}
$$

After applying these transformations, we find

$$
E_{r}^{\prime}=\frac{\gamma q_{0}}{2 \pi \epsilon_{0} r}
$$

And

$$
B_{\theta}^{\prime}=\gamma\left(-\beta E_{r}\right)=-\beta \gamma \frac{q_{0}}{2 \pi \epsilon_{0} r}
$$

The other components vanish by symmetry or explicit calculations whichever you prefer. $E_{\theta}^{\prime}=0, E_{z}^{\prime}=0, B_{r}^{\prime}=0$, and $B_{z}^{\prime}=0$.

Now, I switch rather abruptly to S.I. so that I can compare my results to Griffiths. We have

$$
E_{r}^{\prime}=\frac{\gamma q_{0}}{2 \pi \epsilon_{0} r}
$$

And

$$
B_{\theta}^{\prime}=-\frac{1}{c} \beta \gamma \frac{q_{0}}{2 \pi \epsilon_{0} r}
$$

Fortunately, these compare well to Griffiths' results. It makes sense that $\beta$ is negative if you look at the diagrams which I should scan someday.
b. What are the charge and current densities associated with the wire in its rest frame? In the laboratory?
In $K$, we have

$$
J^{4}=\left(\begin{array}{c}
c \rho \\
J_{z} \\
J_{r} \\
J_{\theta}
\end{array}\right)=\left(\begin{array}{c}
\frac{c}{2 \pi} \frac{q_{0}}{r} \delta(r) \\
0 \\
0 \\
0
\end{array}\right)
$$

And the appropriate Lorentz transformation also can be written in matrix form.

$$
\mathbf{L}=\left(\begin{array}{cccc}
\gamma & -\beta \gamma & 0 & 0 \\
-\beta \gamma & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right)
$$

To transform into the $K^{\prime}$ frame is easy: $J^{4 \prime}=\mathbf{L} J^{4}$. Doing the matrix math gives

$$
\vec{J}^{4 \prime}=J^{4 \prime}=\left(\begin{array}{c}
c \rho^{\prime} \\
J_{z}^{\prime} \\
J_{r}^{\prime} \\
J_{\theta}^{\prime \prime}
\end{array}\right)=\left(\begin{array}{c}
\frac{c \gamma}{2 \pi} \frac{q_{0}}{r} \delta(r) \\
-\frac{c \beta \gamma}{2 \pi} \frac{q}{r} \\
r \\
0
\end{array}\right)
$$

c. From the laboratory charge and current densities, calculate directly the electric and magnetic fields in the laboratory. Compare with the results of part a.
For $K^{\prime}$, we have found $\vec{J}^{4 \prime}$. From the results in part b, we deduce $\lambda^{\prime}=\gamma \lambda$ and $\overrightarrow{J^{\prime}}=-\beta \gamma c q_{0} \delta(r) \hat{z}=-\frac{\beta \gamma q_{0}}{c \mu_{0} \epsilon_{0}} \delta(r) \hat{z}$. For E ,

$$
E_{r}^{\prime}=\frac{\lambda^{\prime}}{2 \pi \epsilon_{0} r}=\frac{\gamma q_{0}}{2 \pi \epsilon_{0} r}
$$

For B,

$$
B_{\theta}^{\prime}=\frac{\mu_{0} J_{z}^{\prime}}{2 \pi r}=\frac{\mu_{0}}{2 \pi r}\left(\frac{-\beta \gamma q_{0}}{c \mu_{0} \epsilon_{0}}\right)=-\frac{1}{c} \beta \gamma \frac{q_{0}}{2 \pi \epsilon_{0} r}
$$

Where I have used the relation, $c=\frac{1}{c \mu_{0} \epsilon_{0}}$. Once again by symmetry, $E_{\theta}=$ $0, E_{z}=0, B_{r}=0$, and $B_{z}=0$.

## Problem 11.15

In a certain reference frame a static, uniform, electric field $E_{0}$ is parallel to the $x$ axis, and a static, uniform, magnetic induction $B_{0}=2 E_{0}$ lies in the $x-y$ plane, making an angle $\theta$ with the axis. Determine the relative velocity of $a$ reference frame in which the electric and magnetic fields are parallel. What are the fields in that frame for $\theta \ll 1$ and $\theta \rightarrow \frac{\pi}{2}$ ?
In $S$ : Given: $\vec{E}=E_{0} \hat{x},|B|=2 E_{0}, \vec{B}$ lies in the $x-y$ plane.
Find an $S^{\prime}$ frame where $E$ and $B$ are parallel. That is a frame where $\vec{E}^{\prime} \times \vec{B}^{\prime}=$ 0 . From Jackson, we have the field transformation.

$$
\begin{aligned}
\vec{E}^{\prime} & =\gamma(\vec{E}+\beta \times \vec{B})-\frac{\gamma^{2}}{\gamma+1} \beta(\beta \cdot \vec{E}) \\
\vec{B}^{\prime} & =\gamma(\vec{B}-\beta \times \vec{E})-\frac{\gamma^{2}}{\gamma+1} \beta(\beta \cdot \vec{B})
\end{aligned}
$$

I want a frame where $\overrightarrow{E^{\prime}} \times B^{\prime}=0$; in all its glory, this can be written

$$
\begin{array}{r}
\frac{1}{\gamma^{2}} \vec{E}^{\prime} \times B^{\prime}= \\
\vec{E} \times \vec{B}-\vec{E} \times(\beta \times \vec{E})+(\beta \times \vec{B}) \times \vec{B}+(\beta \times \vec{B}) \times(\beta \times \vec{E}) \\
+\frac{\gamma^{2}}{\gamma+1}(\beta \times \beta)(\beta \cdot \vec{E})(\beta \cdot \vec{B}) \\
-\frac{\gamma}{\gamma+1}[(\vec{E} \times \beta)(\beta \cdot \vec{B})+(\beta \times \vec{B}) \times \beta(\beta \cdot \vec{B})(\beta \times \vec{B})(\beta \cdot \vec{E})-\beta \times(\beta \times \vec{E})(\beta \cdot \vec{E})] \\
=0
\end{array}
$$

To simplify I must use the following identities:
1.

$$
\beta \times \beta=0
$$

2. 

$$
\vec{E} \times(\beta \times \vec{E})=\beta\left(\vec{E}^{2}\right)-\vec{E}(\beta \cdot \vec{E})
$$

3. 

$$
(\beta \times \vec{B}) \times \vec{B}=-\vec{B} \times(\beta \times \vec{B})=-\beta\left(\vec{B}^{2}\right)+\vec{B}(\beta \cdot \vec{B})
$$

4. 

$$
(\beta \times \vec{B}) \times \beta=-\beta \times(\beta \times \vec{B})=-\beta(\vec{B} \cdot \beta)-\vec{B}\left(\beta^{2}\right)
$$

5. 

$$
\beta \times(\beta \times \vec{E})=\beta(\beta \cdot \vec{E})-\vec{E}\left(\beta^{2}\right)
$$

6. 

$$
(\beta \times \vec{B}) \times(\beta \times \vec{E})=\beta \cdot(\vec{B} \times \vec{E}) \beta-[\beta \cdot(\beta \times \vec{B})] \vec{E}=[\beta \cdot(\vec{B} \times \vec{E})] \beta
$$

Using these relationships, I have

$$
\begin{array}{r}
\vec{E} \times \vec{B}+[\beta \cdot(\vec{B} \times \vec{E})] \beta-\left(\vec{E}^{2}+\vec{B}^{2}\right) \beta+(\beta \cdot \vec{E}) \vec{E}+(\beta \cdot \vec{B}) \vec{B} \\
-\frac{\gamma}{\gamma+1}[-(\beta \times \vec{E})(\beta \cdot \vec{B})+(\beta \times \vec{B})(\beta \cdot \vec{E})] \\
+\frac{\gamma}{\gamma+1}\left[(\beta \cdot \vec{B})\left[-\beta(\beta \cdot \vec{B})+\vec{B} \beta^{2}\right]+(\beta \cdot \vec{E})\left[\beta(\beta \cdot \vec{E})-\vec{E} \beta^{2}\right]\right] \\
=0
\end{array}
$$

This is still exceptionally complicated.
There is a whole plane of possible Lorentz transformations which bring us to a frame where $E$ and $B$ are perpendicular. I won't bother to show this general relationship because Jackson only asks for $a$ frame. That means one! I'll choose the particularly simple case where $\beta$ is along the $z$ axis. In this case $\beta \cdot \vec{E}=0$ and $\beta \cdot \vec{B}=0$. The equation reduces to

$$
(\vec{E} \times \vec{B}) \hat{z}+\beta^{2}(\vec{E} \times \vec{B}) \hat{z}-\beta\left(\vec{E}^{2}+\vec{B}^{2}\right) \hat{z}=0
$$

Upon rearrangement,

$$
\frac{\vec{\beta}}{1+\beta^{2}}=\frac{|\vec{E} \times \vec{B}|}{\vec{E}^{2}+\vec{B}^{2}} \hat{z}
$$

Since $|\vec{E} \times \vec{B}|=2 E_{0}^{2} \sin \theta$ and $\vec{E}^{2}+\vec{B}^{2}=5 \vec{E}^{2}$,

$$
\frac{\beta}{1+\beta^{2}}=\frac{2}{5} \sin \theta
$$

For $\theta \ll 1, \sin \theta \approx \theta$. We can't choose $\beta>1$; it follows that $\beta \approx \frac{2}{5} \theta$ and $\gamma=\sqrt{1-\frac{4}{25} \theta^{2}}$.

$$
E^{\prime}=\gamma(E+\beta \times B)+0=\left(1-\frac{4}{5} \theta^{2}\right)^{-\frac{1}{2}}\left[E_{0}+\frac{4}{5} E_{0} \theta\right]
$$

Take advantage of the invariant quantity $\vec{E} \cdot \vec{B}=\overrightarrow{E^{\prime}} \cdot \vec{B}^{\prime}$.

$$
\vec{E} \cdot \vec{B} \approx 2 E_{0}^{2}=E_{0}\left(1-\frac{4}{25} \theta^{2}\right)^{-\frac{1}{2}}\left(1+\frac{4}{5} \theta\right) B^{\prime} \sin \frac{\pi}{2}
$$

So

$$
B^{\prime}=2 \frac{\sqrt{1-\frac{4}{25} \theta^{2}}}{1-\frac{4}{5} \theta} E_{0}
$$

As $\theta \rightarrow 0$, we get $|E|=E_{0}$ and $|B|=2 E_{0}$ as expected. For $\theta \rightarrow \frac{\pi}{2}$, we the sin term goes to 1 and we get a quadratic equation.

$$
\frac{\beta^{2}}{1+\beta^{2}}=\frac{2}{5} \sin \frac{\pi}{2}=\frac{2}{5} \rightarrow \beta^{2}-\frac{5}{2} \beta+1=0
$$

which has two roots, 2 and $\frac{1}{2}$. The first is clearly unreasonable because $\beta>1$ means that the frame velocity exceeds the speed of light. Good luck getting into that reference frame.
Therefore, $|\beta|=\frac{1}{2}$. In this case, $\vec{E}^{\prime} \rightarrow 0$ and $\vec{B}^{\prime} \rightarrow \sqrt{3} E_{0}$.

## Problem 11.22

The presence in the universe of an apparently uniform sea of blackbody radiation at a temperature of roughly 3 K gives one mechanism for an upper limit on the energies of photons that have traveled an appreciable distance since their creation. Photon-photon collisions can result in the creation of charges particle and its antiparticle (pair creation) if there is sufficient energy in the center of mass of the two photons. The lowest threshold and the largest cross section occurs for an electron-positron pair.
Take $c=1$. At 3 K , the typical energy $E_{1}$ for a background photon is $2.5 \times 10^{-4} \mathrm{eV}$. Assume the momentum for the background photon and the incident photon are anti-parallel. Exploit conservation of energy and the Lorentz invariant properties of four vectors squared.

$$
\binom{2 m_{e}}{0}^{2}=\binom{E_{2}+E_{1}}{p_{2}+p_{1}}^{2}
$$

So

$$
4 m_{e}^{2}=E_{1}^{2}+E_{2}^{2}+2 E_{1} E_{2}-\left(p_{1}^{2}+p_{2}^{2}+2 p_{1} \cdot p_{2}\right)
$$

But $\left|p_{i}\right|=E_{i}$ for photons because they have no mass. Since $p_{1}$ and $p_{2}$ are anti-parallel $p_{1} \cdot p_{2}=-\left|p_{1}\right|\left|p_{2}\right|$. So

$$
4 m_{e}^{2}=E_{1}^{2}+E_{2}^{2}+2 E_{1} E_{2}-\left(E_{1}^{2}+E_{2}^{2}-2 E_{1} E_{2}\right)
$$

Which can easily be rearranged to give

$$
E_{2}=\frac{m_{e}^{2}}{E_{1}}
$$

a. Taking the energy of a typical 3 K photon to be $E=2.5 \times 10^{-4}$ eV , calculate the energy for an incident photon such that their energy just sufficient to make an electron-positron pair. For photons with energies larger than this threshold value, the cross section increases to a maximum of the order of $\left(\frac{e^{2}}{m c^{2}}\right)^{2}$ and then decreases slowly at higher energies. This interaction is one mechanism for the disappearance of such photons as they travel cosmological distance. $E_{2}=1.044 \times 10^{15} \mathrm{eV}$ or $1.67 \times 10^{-4}$ joules.
b. There is some evidence for a diffuse x-ray background with photons having energies of several hundred volts or more. Beyond 1 keV the spectrum falls as $E^{-n}$ with $n \simeq 1.5$. Repeat the calculation of the threshold incident energy, assuming that the energy of the photon in the sea is 500 eV .
Assume $E_{1}=500 \mathrm{eV}$ or $0.5 \mathrm{KeV} . E_{2}=5.22 \times 10^{8} \mathrm{eV}$.
An aside: From thermodynamics, we expect pair production effect to become significant when $k_{b} T \sim m_{e} c^{2}$. This corresponds to a temperature of about $10^{9} \mathrm{~K}$. So we should expect that $E_{2}$ must be very high if $E_{1}$ is very low, on the order of a few or even a few hundred thousand Kelvin, as it is in part a and then $b$.


## Problem 11.23

In a collision process a particle of mass $m_{2}$, at rest in the laboratory, is struck by a particle of mass $m_{1}$, momentum $\vec{p}_{L A B}$ and total energy $E_{L A B}$. In the collision the two initial particles are transformed into two others $m_{3}$ and $m_{4}$. The configurations of the momentum vectors in the center of momentum (cm) frame and the laboratory frame are shown in the figure.

$$
\wp_{1}+\wp_{2} \rightarrow \wp_{3}+\wp_{4}
$$

a. Use invariant scalar products to find the square of the total energy $W$ in the cm frame and to find the cm 3 -momentum.
I am going to take advantage of the fact that the product of two Lorentz four vectors is invariant or the same in all Lorentz frames. A prime denotes that a quantity is measured in the center of momentum frame. no prime means that the quantity is measured in the lab frame. Sometimes, I get a bit carried away and use both subscripts and prime

$$
W^{2}=\binom{E_{1}+m_{2}}{\overrightarrow{p_{1}}}_{l a b}^{2}=E_{1}^{2}+m_{2}^{2}+2 E_{1} m_{2}-\left|\overrightarrow{p_{1}}\right|^{2}
$$

but $E_{i}^{2}-p_{i}^{2}=m_{i}^{2}$ consequently

$$
W^{2}=m_{1}^{2}+m_{2}^{2}+2 E_{1} m_{2}
$$

And in the center of momentum frame,

$$
W^{\prime 2}=\binom{E_{1}^{\prime}+E_{2}^{\prime}}{0}_{C M}^{2} \rightarrow W^{\prime}=E_{1}^{\prime}+E_{2}^{\prime}
$$

So $W^{\prime}$ is the total energy in the center of mass frame. $W^{2}$ is an invariant scalar so $W^{\prime 2}=W^{2}$.
$p^{\prime}=\beta_{2}^{\prime} \gamma_{2}^{\prime} m_{2}$, but $\left|\overrightarrow{p_{2}}\right|=0$ because this particle is at rest. Therefore, in the center of momentum frame, particle two's velocity will be $-\beta_{C M}$ and $\gamma=\gamma_{C M}$. From part b of this problem or through gruesome algebra, we have results for $\beta_{C M}$ and $\gamma_{C M}$. Thus,

$$
p_{C M}^{\prime}=-\beta_{C M}^{\prime} \gamma_{C M}^{\prime} m_{2}=\frac{p_{l a b}}{m_{2}+E_{l a b}} \frac{m_{2}+E_{l a b}}{W} m_{2}=\frac{m_{2} p_{l a b}}{W}
$$

Or if you the gratuitous details of the algebraic approach in lingua latina: Ex principis invarianta quanta habemus

$$
\wp_{l a b}^{2}=\wp_{C M}^{2} \rightarrow\left(E_{l a b}+m_{2}\right)^{2}-p_{l a b}^{2}=\left(E_{1}^{\prime}+E_{2}^{\prime}\right)^{2}=E_{1}^{\prime 2}+E_{2}^{\prime 2}+2 E_{1}^{\prime} E_{2}^{\prime}
$$

Eo quod $E_{i}^{\prime}=\sqrt{m_{1}^{2}+p_{i}^{\prime 2}}$, sequitur

$$
\left(E_{l a b}+m_{2}\right)^{2}-p_{l a b}^{2}=m_{1}^{2}+m_{2}^{2}+2 p^{\prime 2}+2 E_{1}^{\prime} E_{2}^{\prime}
$$

Torqutum

$$
2 m_{2} E_{l a b}=2 p^{\prime 2}+2 E_{1}^{\prime} E_{2}^{\prime} \rightarrow 2 m_{2} E_{l a b}-2 p^{\prime 2}=2 E_{1}^{\prime} E_{2}^{\prime}
$$

Atque

$$
4 m_{2}^{2} E_{l a b}^{2}-8 p^{\prime 2} m_{2} E_{l a b}+4 p^{\prime 2}=4 E_{1}^{2} E_{2}^{2}
$$

Quoniam $E_{i}^{\prime}=\sqrt{m_{1}^{2}+p_{i}^{\prime 2}}$;

$$
4 m_{2}^{2}\left(m_{1}^{2}+p_{l a b}^{2}\right)-8 p^{\prime 2} m_{2} E_{l a b}+4 p^{\prime 4}=4\left(m_{1}^{2}+p_{1}^{\prime 2}\right)\left(m_{2}^{2}+p_{2}^{\prime 2}\right)
$$

Et

$$
m_{2}^{2} p^{\prime 2}-2 m_{2} p^{\prime 2} E_{l a b}=m_{1}^{2} p^{\prime 2}+m_{2}^{2} p^{\prime 2}
$$

Mota posita litterarum

$$
m_{1} p^{\prime 2}+m_{2} p^{\prime 2}+2 m_{2} E_{l a b} p^{\prime 2}=m_{2}^{2} p^{\prime 2}
$$

igitur

$$
W^{2} p^{\prime 2}=m_{2}^{2} p_{l a b}^{2}
$$

tandem

$$
p^{\prime 2}=\frac{m_{2}^{2} p_{l a b}^{2}}{W^{2}}
$$

atque

$$
p^{\prime}=\frac{m_{2} p_{l a b}}{W}
$$

b. Find the Lorentz transformation parameters $\beta_{C M}$ and $\gamma_{C M}$ describing the velocity of the cm frame.
The definition of $\beta_{C M}$ :

$$
\beta_{C M}=\frac{\sum p_{l a b}}{\sum E_{l a b}}=\frac{p_{1}}{E_{1}+m_{2}}
$$

Notice that in the center of momentum frame $\beta_{C M}=0$ as expected since $\beta_{C M}$ describes the velocity of the center of momentum frame relative to the frame in which the momentum and energies are measured.

$$
\gamma_{C M}=\frac{1}{\sqrt{1-\beta^{2}}}=\left[1-\left(\frac{p_{1}}{E_{1}+m_{2}}\right)^{2}\right]^{-\frac{1}{2}}=\frac{E_{1}+m_{2}}{\sqrt{\left(E_{1}+m_{2}\right)^{2}-p_{1}^{2}}}
$$

but $\left(E_{1}+m_{2}\right)^{2}-p_{1}^{2}=E_{1}^{2}+m_{2}^{2}+2 E_{1} m_{2}-p_{1}^{2}$ and $E_{1}^{2}-p_{1}^{2}=m_{1}^{2}$ so the terms in the square root are just $m_{1}^{2}+m_{2}^{2}+2 E_{1} m_{2}=W^{2}$ and

$$
\gamma_{C M}=\frac{E_{1}+m_{2}}{W}
$$

## c. Take the non-relativistic limit of the results in part a and b.

Start with $W^{2}=m_{1}^{2}+m_{2}^{2}+2 E_{1} m_{2}=\left(m_{1}+m_{2}\right)^{2}-2 m_{1} m_{2}+2 E_{1} m_{2}$. Define $T=E_{1}-m_{1}$.

$$
W^{2}=\left(m_{1}+m_{2}\right)^{2}+2 m_{2} T
$$

Write this in a suggestive form.

$$
W=\left(m_{1}+m_{2}\right) \sqrt{1+\frac{2 m_{2} T}{\left(m_{1}+m_{2}\right)^{2}}}
$$

Since $2 m_{2} T \ll m_{1}+m_{2}$, we can use $\sqrt{1+x} \simeq 1+\frac{1}{2} x$.

$$
W \simeq\left(m_{1}+m_{2}\right)\left(1+\frac{m_{2} T}{\left(m_{1}+m_{2}\right)^{2}}\right)=m_{1}+m_{2}+\frac{m_{2}}{m_{1}+m_{2}} T
$$

where $T=E_{1}-m_{1}$. For $v \ll c$ the usual expansion applies $E_{1}=m_{1}+$ $\frac{p^{2}}{2 m_{1}}+O\left(p^{4}\right)$. So $T=m_{1}+\frac{p_{1}^{2}}{2 m_{1}}+\ldots-m_{1}$.

$$
W \simeq m_{1}+m_{2}+\left(\frac{m_{2}}{m_{1}+m_{2}}\right) \frac{p_{1}^{2}}{2 m_{1}}
$$

And $W^{-1} \simeq\left(m_{1}+m_{2}\right)^{-1}$ because the final term in $W$ is so small. Using this we have

$$
p^{\prime}=\frac{m_{2} p_{1}}{W}=\left(\frac{m_{2}}{m_{1}+m_{2}}\right) p_{1}
$$

And

$$
\beta_{C M}=\frac{p_{1}}{m_{2}+E_{1}}=\frac{p_{1}}{m_{1}+m_{2}}
$$

Where I have ignored first order and higher corrections to $E_{1}$, i.e. $E_{1} \simeq m_{1}$.

## Problem 11.26

a. In an elastic scattering process the incident particle imparts energy to the stationary target. The energy $\Delta E$ lost by the incident particle appears as recoil kinetic energy of the target. In the notation of Problem 11.23, $m_{3}=m_{2}$ and $m_{4}=m_{2}$, while $\Delta E=t_{4}=$ $E_{4}-m_{4}$.
a. Show that $\Delta E$ can be expressed in three different ways. $Q^{2}=$ $-\left(\wp_{1}-\wp_{3}\right)^{2}$ is the Lorentz invariant momentum transfer (squared).
The diagrams appropriate to this problem are shown for Jackson problem $11.23 m_{1}$ is the incident particle's mass before the collision, and $m_{2}$ is the struck particle's mass before the collision. $m_{3}$ and $m_{4}$ designate the respective particle masses after the collision. For an elastic collision, $m_{3}=m_{1}$ and $m_{4}=m_{2}$. The unprimed quantities are in the lab frame while the primed quantities are in the center of momentum frame. Note also that $p_{l a b}=p_{1}$ because $p_{2}=0$, the other particle is at rest initially in the lab.
For the first part of this problem, I wanted to do things the brute force way to give you some idea of the kind of awkward algebra involved. But as is typically the case, the algebra ${ }^{6}$ "compelled" me not to take this route.
For the first relationship, I'll use an elegant approach which takes advantage of the invariant properties of four-vectors squared. First of all, consider the scalar product in the center of momentum frame. Note $E_{2}^{\prime}=E_{4}^{\prime}$ because of conservation of momentum.

$$
\wp_{2}^{\prime} \wp_{4}^{\prime}=E_{2}^{\prime} E_{4}^{\prime}-p q \cos \theta^{\prime}=m_{2}^{2}+p_{2}^{\prime 2}-p_{2}^{\prime 2} \cos \theta^{\prime}=m_{2}^{2}+p_{2}^{2}\left(1-\cos \theta^{\prime}\right)
$$

In the lab,

$$
\wp_{2} \wp_{4}=m_{2} E_{4}-0=m_{2}^{2}+m_{2} \Delta E
$$

Using $E_{4}=\Delta E+m_{2}$. Because of Lorentz invariance, $\wp_{2}^{\prime} \wp_{4}^{\prime}=\wp_{2} \wp_{4}$, we get

$$
\Delta E=\frac{p_{2}^{2}}{m_{2}}\left(1-\cos \theta^{\prime}\right)
$$

For the second relationship,

$$
\wp_{1}+\wp_{2}=\wp_{3}+\wp_{4} \rightarrow \wp_{1}-\wp_{4}=\wp_{3}-\wp_{2}
$$

[^5]Square it.

$$
\left(\wp_{1}-\wp_{4}\right)^{2}=\left(\wp_{3}-\wp_{2}\right)^{2}
$$

Multiply out the squares. Don't forget $\wp^{2}=m^{2}$.

$$
m_{1}^{2}+m_{4}^{2}-2\left(E_{1} E_{4}-p_{1} p_{4} \cos \Theta^{\prime}\right)=m_{3}^{2}+m_{2}^{2}-2\left(E_{2} E_{3}-p_{2} p_{3}\right)
$$

But $m_{2}$ is at rest before the collision so $p_{2}=0$ and $E_{2}=m_{2}$. The collision is elastic, that is $m_{1}=m_{3}$ and $m_{2}=m_{4}$, so we get

$$
E_{1} E_{4}-p_{1} p_{4} \cos \Theta^{\prime}=m_{2} E_{3} \rightarrow p_{1} p_{4} \cos \Theta^{\prime}=E_{1} E_{4}-m_{2} E_{3}
$$

As before $E_{4}=\Delta E+m_{2}$ so $E_{3}=E_{1}-\Delta E$.

$$
p_{1} p_{4} \cos \Theta^{\prime}=E_{1}\left(\Delta E+m_{2}\right)-m_{2}\left(E_{1}-\Delta E\right)=\Delta E\left(E_{1}+m_{2}\right)
$$

Square this, and substitute the total center of momentum energy, $W^{2}=$ $\left(E_{1}+m_{2}\right)^{2}-p_{1}^{2}$.

$$
p_{1}^{2} p_{4}^{2} \cos ^{2} \Theta^{\prime}=\Delta E^{2}\left(W^{2}+p_{1}^{2}\right)
$$

Play around with some algebra.

$$
p_{1}^{2}\left(E_{4}^{2}-m_{4}^{2}\right) \cos ^{2} \Theta^{\prime}=p_{1}^{2}\left(\Delta E^{2}+2 m_{2} \Delta E\right) \cos ^{2} \Theta^{\prime}=\Delta E^{2}\left(W^{2}+p_{1}^{2}\right)
$$

Divide by $\Delta E$.

$$
2 m_{2} p_{1}^{2} \cos ^{2} \Theta^{\prime}=\Delta E\left(W^{2}+p_{1}^{2}-p_{1}^{2} \cos ^{2} \Theta^{\prime}\right)=\Delta E\left(W^{2}+p_{1}^{2} \sin ^{2} \Theta^{\prime}\right)
$$

Solve for $\Delta E$.

$$
\Delta E=\frac{2 m_{2} p_{1}^{2} \cos ^{2} \Theta^{\prime}}{W^{2}+p_{1}^{2} \sin ^{2} \Theta^{\prime}}
$$

For the third relation, we need $Q^{2} . Q^{2}$ is defined as follows

$$
Q^{2}=-\left(\wp_{1}-\wp_{3}\right)^{2}=\left(p_{1}-p_{3}\right)^{2}-\left(E_{1}-E_{2}\right)^{2}
$$

In the center of momentum frame, $\left|p_{1}^{\prime}\right|=\left|p_{3}^{\prime}\right|=p^{\prime}$ so

$$
E_{1}^{\prime}-E_{3}^{\prime}=\sqrt{m^{2}+p_{1}^{\prime 2}}-\sqrt{m^{2}+p_{3}^{\prime 2}}=0
$$

And

$$
\left.\begin{array}{rl}
\left(p_{1}^{\prime}-p_{3}^{\prime}\right)^{2}= & p_{1}^{\prime 2}+p_{3}^{\prime 2}
\end{array}\right) 2 p_{1}^{\prime} p_{3}^{\prime} \cos \theta^{\prime} \rightarrow \quad . ~\left(p_{1}^{\prime}-p_{3}^{\prime}\right)^{2}=2 p^{\prime 2}\left(1-\cos \theta^{\prime}\right) ~ \$
$$

Thus,

$$
Q^{2}=2 p^{\prime 2}\left(1-\cos \theta^{\prime}\right)
$$

Use the first relation, $\Delta E=\frac{p^{\prime 2}}{m_{2}}\left(1-\cos \theta^{\prime}\right)$, from problem 11.23. Substitute $Q^{2}$ and get

$$
\Delta E=\frac{Q^{2}}{2 m_{2}}
$$

b. Show that for charged particles other than electrons incident on stationary electrons ( $m_{1} \gg m_{2}$ ) the maximum energy loss is approximately

$$
\Delta E_{M a x} \simeq 2 m_{e} \gamma^{2} \beta^{2}
$$

where $\gamma$ and $\beta$ are characteristic of the incident particle and $\gamma \ll$ $\frac{m_{1}}{m_{e}}$. Give this result a simple interpretation by considering the relevant collision in the rest frame of the incident particle and then transforming back to the laboratory.
For $m_{1} \gg m_{2}$, we start with the first expression for the energy change.

$$
\Delta E=\frac{m_{2} p_{1}^{2}}{W^{2}}\left(1-\cos \theta^{\prime}\right)
$$

This attains its greatest value when $\theta^{\prime}=\pi$ and $1-\cos \theta^{\prime}=2$, then

$$
\Delta E_{M a x}=\frac{2 m_{2} p_{1}^{2}}{W^{2}}=\frac{2 m_{e} \gamma^{2} \beta^{2} m_{1}^{2}}{W^{2}}
$$

$W^{2}=m_{1}^{2}+m_{e}^{2}+2 m_{e} E_{1}$ can be written in a suggestive manner.

$$
W^{2}=m_{1}^{2}\left(1+\frac{m_{e}^{2}}{m_{1}^{2}}+2 \frac{m_{e}}{m_{1}} \frac{\gamma}{\frac{m_{1}}{m_{e}}}\right)
$$

Because $m_{1} \gg m_{e}$, the second term in the brackets is small and can be ignored to first order. Because $\gamma \ll \frac{m_{1}}{m_{e}}$, the third term is similarly small and can be ignored. So we have $W^{2} \simeq m_{1}^{2}$. We conclude

$$
\Delta E_{M a x} \simeq 2 m_{e} \gamma^{2} \beta^{2}
$$

When Jackson asks for a simple interpretation, I'm not sure what the Hell he wants. His question is vague. Maybe, he wants us to make some statement about $m_{1}$ being almost stationary. You'll have to bullshit your way through this.
c. For electron-electron collisions, find the maximum energy transfer.
For electron collisions, $m_{1}=m_{2}=m_{e}$. Use the formula for $\Delta E$ from the beginning of part b, but substitute in $W$ explicitly.

$$
\begin{aligned}
\Delta E_{M a x} & =\frac{2 m_{2} p_{1}^{2}}{2 m_{2} E_{1}+m_{1}^{2}+m_{2}^{2}}=\frac{m_{e} p_{1}^{2}}{m_{e} E_{1}+m_{e}^{2}} \\
& =\frac{p^{2}}{E+m}=\frac{\gamma^{2} \beta^{2} m^{2}}{\gamma m+m}=\left(\frac{\gamma^{2} \beta^{2}}{\gamma+1}\right) m_{e}
\end{aligned}
$$

With a little trivial algebra, we find $\beta^{2}=\frac{\gamma^{2}-1}{\gamma^{2}}$. Substitute in for $\beta$ and get the desired result.

$$
\Delta E_{M a x}=(\gamma-1) m_{e}
$$

## Problem 12.3

A particle with mass $m$ and charge $e$ moves in a uniform, static, electric field field $\vec{E}_{0}$.
a. Solve for the velocity and position of the particle as explicit functions of time, assuming that the initial velocity $\vec{v}_{0}$ was perpendicular to the electric field.
Take $\vec{F}=e \vec{E}_{0} \hat{x}$. Since the initial velocity is non-vanishing but perpendicular to the field, we have $\overrightarrow{v_{0}}=v_{0} \hat{y}$. The relativistic force law is $\frac{d \rho_{i}}{d t}=F_{i}$, so we have two equations which must be satisfied.

$$
\frac{d}{d t}\left(\frac{m v \hat{x}}{\sqrt{1-\frac{v^{2}}{c^{2}}}}\right)=e E_{0} \hat{x}
$$

And

$$
\frac{d}{d t}\left(\frac{m v \hat{y}}{\sqrt{1-\frac{v^{2}}{c^{2}}}}\right)=0
$$

Integrate these.

$$
\frac{m v_{x}}{\sqrt{1-\frac{v^{2}}{c^{2}}}}=e E_{0} t+C_{1}
$$

$C_{1}=0$ because the initial $x$ velocity is zero.

$$
\frac{m v_{y}}{\sqrt{1-\frac{v^{2}}{c^{2}}}}=C_{2}
$$

To find $C_{2}$, we must invoke initial conditions. At time initial, $v_{y}=v_{0}$ and $v^{2}=v_{0}^{2}$. Then,

$$
\frac{m v_{0}}{\sqrt{1-\frac{v_{0}^{2}}{c^{2}}}}=C_{2} \rightarrow C_{2}=\frac{m v_{0} c}{\sqrt{c^{2}-v_{0}^{2}}}
$$

We can get $v_{x}$ and $v_{y}$ as functions of time.

$$
\begin{gathered}
v_{x}^{2}=\frac{e^{2} E_{0}^{2} t^{2}}{m^{2}}\left(1-v_{x}^{2}-v_{y}^{2}\right) \\
v_{y}^{2}=\frac{C_{2}^{2}}{m^{2}}\left(1-v_{x}^{2}-v_{y}^{2}\right)
\end{gathered}
$$

Dividing these two, we get a relationship between $v_{x}$ and $v_{y}$.

$$
\frac{v_{x}^{2}}{v_{y}^{2}}=\frac{e^{2} E_{0}^{2} t^{2}}{A^{2}}
$$

Now, solve for $v_{x}$ and $v_{y}$.

$$
\begin{aligned}
& v_{x}^{2}=\frac{c^{2} e^{2} E_{0}^{2} t^{2}}{c^{2} m^{2}+e^{2} E_{0}^{2} t^{2}+C_{2}^{2}} \\
& v_{y}^{2}=\frac{C_{2}^{2}}{c^{2} m^{2}+e^{2} E_{0}^{2} t^{2}+C_{2}^{2}}
\end{aligned}
$$

Define $\gamma_{0}=\left(1-\frac{v_{0}^{2}}{c^{2}}\right)^{-\frac{1}{2}}$ and $a=\frac{q E}{m c}$. We now have separate equations for $v_{x}$ and $v_{y}$.

$$
v_{x}=\frac{c a t}{\sqrt{a^{2} t^{2}+\gamma_{0}^{2}}}
$$

And

$$
v_{y}=\frac{\gamma_{0} v_{0}}{\sqrt{a^{2} t^{2}+\gamma_{0}^{2}}}
$$

These two can be integrated over time to get the expressions for $x(t)$ and $y(t)$.

$$
x(t)=c \int_{0}^{t} \frac{a t^{\prime}}{\sqrt{a^{2} t^{\prime 2}+\gamma_{0}^{2}}} d t^{\prime}=\frac{c}{a}\left[\sqrt{a^{2} t^{2}+\gamma_{0}^{2}}-\gamma_{0}\right]
$$

And

$$
y(t)=\gamma_{0} v_{0} \int_{0}^{t} \frac{1}{\sqrt{a^{2} t^{\prime 2}+\gamma_{0}^{2}}} d t^{\prime}=\frac{\gamma_{0} v_{0}}{a} \ln \left[\frac{\sqrt{a^{2} t^{2}+\gamma_{0}^{2}}+a t}{\gamma_{0}}\right]
$$

## Problem 12.4

It is desired to make an $E \times B$ velocity selector with uniform, static, crossed, electric and magnetic fields over a length $L$. If the entrance and exit slit widths are $\Delta x$, discuss the interval $\Delta u$ of velocities, around the mean value $u=\frac{c E}{B}$, that is transmitted by the device as a function of the mass, the momentum or energy of the incident particles, the field strengths, the length of the selector, and any other relevant variables. Neglect fringing effects at the ends. Base your discussion on the practical facts that $L \approx$ a few meters, $E_{\text {Max }} \approx$ $3 \times 10^{6} \mathrm{~V} / \mathrm{m}, \Delta x \approx 10^{-3}-10^{-4} \mathrm{~m}$, and $u \approx 0.5-0.995 c$.
Consider the design of an $\vec{E} \times \vec{B}$ velocity selector. Take $c=1$ as usual.
In $S, \vec{E}=E \hat{y}, \vec{B}=B \hat{z}, \overrightarrow{u_{0}}=u_{0} \hat{x}$, and $u_{0}=\frac{E}{B}$ where $u_{0}$ is the average selected velocity. The aperture admittance is $\Delta x$, and the length of the selector is $L$. Let $L=u_{0} \bar{t} \rightarrow \bar{t}=\frac{L}{u_{0}} . \bar{t}$ is the average time per particle in the selector.
Go to $S^{\prime}$, a frame moving $\vec{u}=\frac{E}{B} \hat{x}$. We are moving along with the particles as they pass through the selector. Note that in this frame, the following transformations hold:

$$
E^{\prime}=\gamma(E-u B)=\gamma(E-E)=0
$$

And

$$
B^{\prime}=\gamma(B-u E) \hat{z}=\gamma\left(B-\frac{E^{2}}{B}\right) \hat{z}=\frac{\gamma}{B}\left(B^{2}-E^{2}\right) \hat{z}
$$

Which can be further simplified because $\gamma=\left(1-u^{2}\right)^{-\frac{1}{2}}=\left(1-\frac{E^{2}}{B^{2}}\right)^{-\frac{1}{2}}=$ $\frac{B}{\sqrt{B^{2}-E^{2}}}$, so

$$
B^{\prime}=\sqrt{B^{2}-E^{2}} \hat{z}=\frac{B}{\gamma} \hat{z}
$$

Particles which have $\vec{\beta}=u_{0} \hat{x}$ in the lab will be at rest in $S^{\prime}$ and so will be unaffected by the field. Also in $S^{\prime}$, the time it takes for a particle to travel from one aperture to the other is given by $t^{\prime}=\frac{L}{\gamma u}$. (More appropriately, this is the time it takes one aperture to move away and the other one to arrive!) The $\gamma$ comes in because in this frame the selector is moving so the distance is contracted. A particle with non-zero velocity in $S^{\prime}$ will be deflected in an arc. I'll draw this for clarity someday.

$$
\Delta x^{\prime}=r_{0}^{\prime}\left(1-\cos \omega_{B}^{\prime} t^{\prime}\right)
$$

Since $\Delta x^{\prime}$ is perpendicular to $u, \Delta x^{\prime}=\Delta x$. Jackson told us $\omega_{B}^{\prime} \simeq \frac{q B^{\prime}}{m}=\frac{q B}{\gamma m}$ and $r_{0}^{\prime}=\frac{p_{1}^{\prime}}{q B^{\prime}}=\frac{m v^{\prime}}{q B^{\prime}}=\frac{m}{q B^{\prime}} v^{\prime}=\frac{v^{\prime}}{\omega_{B}^{\prime}}$.
We expect the deflection to be small because the aperture we are considering is small. Thus, we approximate $\cos \omega_{B}^{\prime} t^{\prime} \simeq 1-\frac{\left(\omega_{B}^{\prime} t^{\prime}\right)^{2}}{2}$. So

$$
\Delta x^{\prime}=\frac{\omega_{B}^{2} t^{\prime 2}}{2} r_{0}^{\prime}
$$

Get $v^{\prime}$ in terms of the variables we know, namely $\Delta u$ and $u_{0}$. Assuming that $v$ is small compared to $u$, we can use the approximation that $u v \simeq u^{2}$ so that $1-u v \simeq \gamma^{2}$.

$$
v^{\prime}=\frac{v-u}{1-u v}=\gamma^{2} \Delta u
$$

Now, plug in $v^{\prime}$ to the expression for $\Delta x$.

$$
\Delta x=\frac{q^{2} B^{2}}{2 \gamma^{2} m^{2}} \frac{L^{2}}{\gamma^{2} u^{2}} \frac{\gamma m}{q B} \gamma^{2} \Delta u
$$

Simplify and substitute $B=\frac{E}{u}$. We have the following expression for the deflection:

$$
\Delta x=\frac{q E L^{2}}{2 \gamma m u^{3}} \Delta u
$$

With $\gamma u m=p$ and some rearrangement, you should get

$$
\Delta u=\frac{2 p}{q L^{2} E} u^{2} \Delta x
$$

Depending on how you define $\Delta u$ there may be a factor of two missing.

## Problem 12.5

A particle of mass $m$ and charge $e$ moves in the laboratory in crossed, static, uniform, electric and magnetic fields. $\vec{E}$ is parallel to the $x$ axis; $\vec{B}$ is parallel to the $y$ axis.
a. For $|E|<|B|$ make the necessary Lorentz transformation described in Section 12.3 to obtain explicitly parametric equations for the particles trajectory.
$c=1, \vec{E}=E \hat{x}, \vec{B}=B \hat{y}$. Since $|E|<|B|$, we can transform the $E$ field away by choosing a suitable Lorentz frame. Try $\vec{u}=\frac{\vec{E} \times \vec{B}}{B^{2}} \rightarrow \vec{u}=\frac{E}{B} \hat{z}$. The appropriate Lorentz transformations are:

$$
\begin{aligned}
E^{\prime} & =\gamma(E+\beta \times B)-\frac{\gamma^{2}}{\gamma+1} \beta(\beta \cdot E) \\
B^{\prime} & =\gamma(B-\beta \times E)-\frac{\gamma^{2}}{\gamma+1} \beta(\beta \cdot B)
\end{aligned}
$$

Note $\beta \cdot E=0$ and $\beta \cdot B=0$. With these, the fields transform as such.

$$
\begin{gathered}
E^{\prime}=\gamma(E+\beta \times B)=\gamma(E-u B) \hat{y}=\gamma\left(E-\frac{E}{B} B\right)=0 \\
B^{\prime}=\gamma(B-\beta \times E)=\gamma(B-u E) \hat{y}=\frac{\gamma}{B}\left(B^{2}-E^{2}\right) \hat{y}
\end{gathered}
$$

But $\gamma=\left(1-\frac{E^{2}}{B^{2}}\right)^{-\frac{1}{2}}=\frac{B}{\sqrt{B^{2}-E^{2}}}$ so

$$
B^{\prime}=\sqrt{B^{2}-E^{2}} \hat{y}
$$

Which can be expressed as in Jackson

$$
B_{p e r p}^{\prime}=\sqrt{\frac{B^{2}-E^{2}}{B^{2}}} \vec{B}
$$

In this frame, we have a particle moving in a uniform static $B$ field. Jackson solved this for us.

$$
\vec{x}(t)=\vec{x}_{0}+v_{\|} t \overrightarrow{\epsilon_{3}}+i a\left(\overrightarrow{\epsilon_{1}}-i \overrightarrow{\epsilon_{2}}\right) e^{-i \omega_{B} t}
$$

Matching initial conditions requires

$$
\vec{x}(t)=v_{y} t \hat{y}+a \cos \omega_{B} t \hat{x}+a \sin \omega_{B} t \hat{z}
$$

where $\omega_{B}=\frac{e B^{\prime}}{\gamma m}$ and $a=\frac{p_{x}^{2}+p_{z}^{2}}{e B}$.
Now, I simply transform back to the lab to get what Jackson wants.

$$
\begin{gathered}
u=-\frac{E}{B} \hat{z} \\
\gamma=\frac{B}{\sqrt{B^{2}-E^{2}}}
\end{gathered}
$$

Jackson only wants parametric equation so I won't bother with the complication that $t=f\left(t^{\prime}\right)$. The equation of motion along the $\hat{y}$ direction is easy because the fields do not accelerate the particle along this direction. The $\hat{x}$ part is unaffected by the Lorentz transformation. The $\hat{z}$ component is not much more difficult. Just multiply by appropriate length contraction $\gamma$ on the $z$ position in $S^{\prime}$ and add an additional term to account for the motion of the frame. The $\gamma$ factor on the latter term is necessary because the time is given in the other frame. The final result is

$$
\vec{x}_{l a b}(t)=a \cos \left(\omega_{B} t\right) \hat{x}+v_{y, t=0} t \hat{y}+\left(\gamma a \sin \left(\omega_{B} t\right)+\gamma u t\right) \hat{z}
$$

b. Repeat the calculation for $|E|>|B|$.

I didn't do.

## Problem 12.14

An alternative Lagrangian density for the electro-magnetic field is

$$
\mathcal{L}=-\frac{1}{8 \pi} \partial_{\alpha} A_{\beta} \partial^{\alpha} A^{\beta}-\frac{1}{c} J_{\alpha} A^{\alpha}
$$

a. Derive the Euler-Lagrange equations of motion? Under what assumption? (Where's the verb in the last sentence?)
The Euler-Lagrange theorem says

$$
\frac{\partial \mathcal{L}}{\partial \phi^{\alpha}}=\partial^{\beta} \frac{\partial \mathcal{L}}{\partial\left(\partial^{\beta} \phi^{\alpha}\right)}
$$

So we have $\frac{\partial \mathcal{L}}{\partial A^{\alpha}}=-\frac{1}{c} J_{\alpha}$ and

$$
\frac{\partial \mathcal{L}}{\partial\left(\partial^{\beta} A^{\alpha}\right)}=-\frac{1}{8 \pi} \frac{\partial}{\partial\left(\partial^{\beta} A^{\alpha}\right)}\left(g_{\sigma \mu} g_{\tau \nu} \partial^{\mu} A^{\nu} \partial^{\sigma} A^{\tau}\right)
$$

Recall that the rule for differentiation is $\frac{\partial}{\partial\left(\partial^{\kappa} A^{\lambda}\right)}\left(\partial^{\eta} A^{\gamma}\right)=\delta_{\kappa \eta} \delta_{\lambda \gamma}$.

$$
\frac{\partial \mathcal{L}}{\partial\left(\partial^{\beta} A^{\alpha}\right)}=-\frac{1}{8 \pi} g_{\sigma \mu} g_{\tau \nu}\left[\delta_{\beta \mu} \delta_{\alpha \nu} \partial^{\mu} A^{\nu}+\delta_{\beta \sigma} \delta_{\alpha \tau} \partial^{\sigma} A^{\tau}\right]
$$

Using the Dirac deltas, we get

$$
\frac{\partial \mathcal{L}}{\partial\left(\partial^{\beta} A^{\alpha}\right)}=-\frac{1}{8 \pi}\left[g_{\sigma \beta} g_{\tau \alpha} \partial^{\beta} A^{\alpha}+g_{\beta \mu} g_{\alpha \nu} \partial^{\beta} A^{\alpha}\right]=-\frac{1}{8 \pi}\left[2 \partial_{\beta} A_{\alpha}\right]
$$

The Euler-Lagrange equation of motion is, in our case,

$$
\begin{equation*}
\partial^{\beta} \partial_{\beta} A_{\alpha}=\frac{4 \pi}{c} J_{\alpha} \tag{9}
\end{equation*}
$$

If we are in the Lorentz gauge, $\partial_{\mu} A^{\mu}=0$, and we can write equation 9 as $\partial^{\beta} F_{\beta \alpha}=\frac{4 \pi}{c} J_{\alpha}$ because $F_{\beta \alpha}=\partial_{\beta} A_{\alpha}-\partial_{\alpha} A_{\beta}=\partial_{\beta} \partial_{\alpha}$. We have the inhomogeneous Maxwell equations!
b. Show explicitly, and with what assumptions, that this Lagrangian density differs from (12.85) by a 4 -divergence. Does this added 4-divergence affect the action or the equations of motion? The other Lagrangian is

$$
\mathcal{L}=-\frac{1}{16 \pi} F_{\alpha \beta} F^{\alpha \beta}-\frac{1}{c} J_{\alpha} A^{\alpha}
$$

Write $F_{\alpha \beta}$ explicitly as $\partial_{\alpha} A_{\beta}-\partial_{\beta} A_{\alpha}$.

$$
\mathcal{L}=-\frac{1}{16 \pi}\left(\partial_{\alpha} A_{\beta}-\partial_{\beta} A_{\alpha}\right)\left(\partial^{\alpha} A^{\beta}-\partial^{\beta} A^{\alpha}\right)-\frac{1}{c} J_{\alpha} A^{\alpha}
$$

The difference between this Lagrangian and the one in part a is

$$
\begin{gathered}
\Delta \mathcal{L}=-\frac{1}{16 \pi}\left[\partial_{\alpha} A_{\beta} \partial^{\alpha} A^{\beta}-\partial_{\beta} A_{\alpha} \partial^{\alpha} A^{\beta}-\partial_{\alpha} A_{\beta} \partial^{\beta} A^{\alpha}+\partial_{\beta} A_{\alpha} \partial^{\beta} A^{\alpha}-2 \partial_{\alpha} A_{\beta} \partial^{\alpha} A^{\beta}\right] \\
\Delta \mathcal{L}=\frac{1}{16 \pi}\left[\partial_{\beta} A_{\alpha} \partial^{\alpha} A^{\beta}+\partial_{\alpha} A_{\beta} \partial^{\beta} A^{\beta}\right]=\frac{1}{8 \pi} \partial_{\alpha} A_{\beta} \partial^{\beta} A^{\alpha}
\end{gathered}
$$

And by using the rule for differentiating a product.

$$
\frac{1}{8 \pi} \partial_{\alpha} A_{\beta} \partial^{\beta} A^{\alpha}=\frac{1}{8 \pi} \partial_{\alpha}\left(A_{\beta} \partial^{\beta} A^{\alpha}\right)-\frac{1}{8 \pi} A_{\beta} \partial^{\beta} \partial_{\alpha} A^{\alpha}
$$

A careful reader will notice that I have switched the order of differentiation on the last term. This is allowed because derivatives commute, i.e. $\left[\partial_{\gamma}, \partial_{\eta}\right]=0$. In the Lorentz gauge, $\partial_{\alpha} A^{\alpha}=0$, and the last term vanishes, $\frac{1}{8 \pi} A_{\beta} \partial^{\beta} \partial_{\alpha} A^{\alpha}=$ 0 . The remaining term, $\frac{1}{8 \pi} \partial_{\alpha}\left(A_{\beta} \partial^{\beta} A^{\alpha}\right)$, is just a four divergence.

## Problem 13.4

a. Taking $\hbar\langle\omega\rangle=12 Z \mathrm{eV}$ in the quantum-mechanical energy-loss formula, calculate the rate of energy loss $9 \mathrm{in} \mathrm{MeV} / \mathrm{cm}$ ) in air at NTP, aluminum, copper, and lead for a proton and a mu meson (muon), each with kinetic energies of 10,100 , and 1000 MeV .
b. Convert your results to energy loss in units of $\mathrm{MeV}\left(\mathrm{cm}^{2} / \mathrm{g}\right)$ and compare the values obtained in different materials. Explain why all the energy losses in $\mathrm{MeV}\left(\mathrm{cm}^{2} / \mathrm{g}\right)$ are within a factor of 2 of each other, whereas the values in $\mathrm{meV} / \mathrm{cm}$ differ greatly.
The quantum mechanical energy loss formula is:

$$
\frac{d E}{d x}=4 \pi N Z \frac{z^{2} e^{4}}{m_{e} c^{2} \beta^{2}}\left[\ln \left(\frac{2 \gamma^{2} \beta^{2} m_{e} c^{2}}{\hbar\langle\omega\rangle}\right)-\beta^{2}\right]
$$

This formula gives results in units of energy per distance. Numerically, $4 \pi \frac{z^{2} e^{4}}{m_{e} c^{2}}=5.1 \times 10^{-25} \mathrm{Mev} \mathrm{cm}^{2}$, and $\frac{2 m_{e} c^{2}}{\hbar\langle\omega\rangle / Z}=\frac{2 m_{e} c^{2}}{12}=8.5 \times 10^{4}$. The $m_{e} c^{2}$ must be given in eV .
Another formula can be constructed which has units of energy times area per mass. I do that by dividing the first result by $\rho$, the density. $\rho$ is equal to $N A m_{\text {nucleon }}$.

$$
\frac{d E}{d x} / \rho=4 \pi \frac{Z}{A m_{n}} \frac{z^{2} e^{4}}{m_{e} c^{2} \beta^{2}}\left[\ln \left(\frac{2 \gamma^{2} \beta^{2} m c^{2}}{\hbar\langle\omega\rangle}\right)-\beta^{2}\right]
$$

$\beta$ and $\gamma$ can be determined for the muon and the electron using the relationship $\beta=\frac{p}{E}, E=T+m, E^{2}=p^{2}+m^{2}$ (These formulas require that I use units so that $c=1$ and $\hbar=1$ ).
Aluminum has $Z=13, Z=27$, and density, $\rho=2.7 \mathrm{gm} / \mathrm{cm}^{3}$. Copper has $Z=29, A=64$, and $\rho=9.0$. Lead has $Z=82, A=208$, and $\rho=11$. For air, we use Nitrogen, $Z=14, A=28$, and $\rho=1.3 \times 10^{-3}$.
The energy loss per densities should be roughly the same because the electron densities are similar if the atomic densities are the same. By dividing out the density, we give out answer in a form that is independent or the atomic density.
Incident Protons with Various Energies. (Energy Loss in Mev/cm)

|  | 10 Mev | 100 MeV | 1000 MeV |
| :---: | :---: | :---: | :---: |
| air | $5 \times 10^{-2}$ | $8 \times 10^{-3}$ | $3 \times 10^{-3}$ |
| Al | 100 | 17 | 5.2 |
| Cu | 310 | 52 | 16 |
| Pb | 330 | 55 | 17 |

Incident Muons with Various Energies. (Energy Loss in Mev/cm)

|  | 10 Mev | 100 MeV | 1000 MeV |
| :---: | :---: | :---: | :---: |
| air | $9 \times 10^{-3}$ | $2.6 \times 10^{-3}$ | $2.7 \times 10^{-3}$ |
| Al | 19 | 5.4 | 5.6 |
| Cu | 58 | 17 | 17 |
| Pb | 61 | 18 | 18 |

Incident Protons. (Energy Loss in Mev cm ${ }^{2} / \mathrm{gm}$ )

|  | 10 Mev | 100 MeV | 1000 MeV |
| :--- | :---: | :---: | :---: |
| air | 37 | 6.1 | 1.9 |
| Al | 37 | 6.3 | 1.9 |
| Cu | 34.8 | 5.8 | 1.8 |
| Pb | 30 | 5.0 | 1.6 |

Incident Muons. (Energy Loss in Mev $\mathrm{cm}^{2} / \mathrm{gm}$ )

|  | 10 Mev | 100 MeV | 1000 MeV |
| :---: | :---: | :---: | :---: |
| air | 6.8 | 2.0 | 2.1 |
| Al | 7.0 | 2.0 | 2.1 |
| Cu | 6.5 | 1.9 | 1.9 |
| Pb | 5.6 | 1.6 | 1.6 |

## Problem 13.9

Assuming that Plexiglas or Lucite has an index of retraction of 1.50 in the visible region, compute the angle of emission of visible Cherenkov radiation for electrons and protons as a function of their kinetic energies in MeV . Determine how many quanta with wavelengths between 4000 and 6000 Angstroms are emitted per centimeter of path in Lucite by a 1 meV electron, a 500 MeV proton, and a 5 GeV proton.
As usual, I'm going to take $c=1$. We are asked to consider the Cherenkov radiation for Plexiglas or Lucite. I think by index of retraction Jackson meant index of refraction, i.e. $n=1.5$. From Jackson 13.50, we have: $\cos \theta_{c}=\frac{1}{\beta \sqrt{\epsilon(\omega)}}=\frac{1}{\beta n}$. The last equality is true because from Jackson 13.47, $v=\frac{c}{\sqrt{\epsilon(\omega)}}$ but also $v=\frac{c}{n}$, so $\sqrt{\epsilon(\omega)}=n$. To solve for $\beta$, use $\beta=\frac{p}{E}$. Since $E=T+m$, this gives us $\beta=\frac{\sqrt{(m+T)^{2}-m^{2}}}{m+T}=\frac{\sqrt{T^{2}+2 T m}}{m+T}$.
To find the number of photons within some energy range emitted per unit length, consult the Particle Physics Data book to find

$$
\frac{d^{2} N}{d \lambda d x}=\frac{-2 \pi \alpha z^{2}}{\lambda^{2}} \sin ^{2} \theta_{c}
$$

This can also be derived from Jackson 13.48.

$$
\frac{d^{2} E}{d x d \omega}=\frac{z^{2} e^{2}}{c^{2}} \omega\left(1-\frac{1}{\beta^{2} n^{2}}\right)
$$

Now, in cgs units, $e^{2}=\alpha \hbar c$, so I can write

$$
\frac{d^{2} E}{d x d \omega}=\frac{z^{2} \alpha \hbar}{c} \omega\left(1-\frac{1}{\beta^{2} n^{2}}\right)
$$

Notice that $\frac{1}{\beta^{2} n^{2}}=\cos ^{2} \theta_{c}$. Thus, the term in parenthesis can be reduced using elementary trigonometric relations to $\sin ^{2} \theta_{c}$. Now, I make a dubious step.

$$
E=N \hbar \omega \rightarrow d^{2} E=-d^{2} N \hbar \omega
$$

So we have

$$
\frac{d^{2} N}{d x d \omega}=\frac{-z^{2} \alpha}{c} \sin ^{2} \theta_{c}
$$

Then, $\omega=\frac{2 \pi c}{\lambda}$ so $d \omega=\frac{-2 \pi c}{\lambda^{2}} d \lambda$. And finally, we get

$$
\frac{d^{2} N}{d x d \lambda}=\frac{2 \pi \alpha z^{2}}{\lambda^{2}} \sin ^{2} \theta_{c}
$$

which is the same as equation .
Integrate over $\lambda$.

$$
\frac{d N}{d x}=\int_{\lambda_{1}}^{\lambda_{2}} \frac{-2 \pi \alpha z^{2}}{\lambda^{2}} \sin ^{2} \theta_{c} d \lambda=2 \pi \alpha z^{2} \sin ^{2} \theta_{c}\left(\frac{\lambda_{2}-\lambda_{1}}{\lambda_{2} \lambda_{1}}\right)
$$

Using $\lambda_{1}=4000\left|\vec{r}-\vec{r}^{\prime}\right| A$ and $\lambda_{2}=6000\left|\vec{r}-\vec{r}^{\prime}\right| A$, we have a numerical expression.

$$
\frac{d N}{d x} \simeq 382.19 \sin ^{2} \theta_{c}
$$

in units of $\mathrm{MeV} / \mathrm{cm} . \theta_{c}$ is related to $n$ and $\beta$ from the results in part a.
I have lot's of cool Maple plots which I plan on including but for now, I'll just give you the final numbers.
For an incident electron with $T=1 \mathrm{MeV}$, the number of Cherenkov photons is about 187. The critical angle is 0.78 rad .
For an incident proton with $T=500 \mathrm{MeV}$, the number of Cherenkov photons is about 79. The critical angle is 0.50 rad .
For an incident proton with $T=5 \mathrm{TeV}$, the number of Cherenkov photons is about 208. The critical angle is 0.83 rad .

## Problem 14.5

A non-relativistic particle of charge $Z q$, mass $m$, and kinetic energy $T$ makes a head on collision with fixed central force field of infinite range. The interaction is repulsive and described by a potential $V(r)$, which becomes greater than $E$ at close distances.
a. Find the total energy radiated.

The total energy for the particle is constant.

$$
\begin{equation*}
E=\frac{m v^{2}}{2}+V(r) \tag{10}
\end{equation*}
$$

At $r_{\text {min }}$, the velocity will vanish and $E=V\left(r_{\text {min }}\right)$.
From Jackson equation 14.21, we have the power radiated per solid angle for an accelerated charge.

$$
\frac{d P}{d \Omega}=\frac{Z^{2} q^{2}}{4 \pi c^{3}}|\dot{v}|^{2} \sin ^{2} \theta
$$

From Newton's second law, $m|\dot{v}|=\left|\frac{d V}{d r}\right|$ so

$$
\frac{d P}{d \Omega}=\frac{Z^{2} q^{2}}{4 \pi c^{3} m^{2}}\left|\frac{d V}{d r}\right|^{2} \sin ^{2} \theta
$$

The total power is $\frac{d P}{d \Omega}$ integrated over all solid angles.

$$
P_{\text {total }}=\int \frac{d P}{d \Omega} d \Omega=\frac{Z^{2} q^{2}}{4 \pi c^{3} m^{2}}\left|\frac{d V}{d r}\right|^{2} \int_{0}^{\pi} \sin ^{2} \theta d \theta \int_{0}^{2 \pi} d \phi
$$

Evaluating the integrals, $\int_{0}^{\pi} \sin ^{2} \theta d \theta=\frac{4}{3}$ and $\int_{0}^{2 \pi} d \phi=2 \pi$ gives

$$
P_{\text {total }}=\frac{2}{3} \frac{Z^{2} q^{2}}{c^{3} m^{2}}\left|\frac{d V}{d r}\right|^{2}
$$

The total work is the power integrated over the entire trip:

$$
W_{\text {total }}=\int P_{\text {total }} d t=2 \times \frac{2}{3} \frac{Z^{2} q^{2}}{c^{3} m^{2}} \int\left|\frac{d V}{d r}\right|^{2} d t
$$

The factor of two comes because the particle radiates as it accelerates to and from the potential. We can solve equation 10 for $v$.

$$
v=\frac{d r}{d t}=\sqrt{\frac{2}{m}\left[V_{\min }-V(r)\right]}
$$

And from this equation, we find, $d t=\frac{d r}{\sqrt{\frac{2}{m}\left[V_{\text {min }}-V(r)\right]}}$. So

$$
W_{\text {total }}=\frac{4}{3} \frac{Z^{2} q^{2}}{c^{3} m^{2}} \int_{0}^{\infty}\left|\frac{d V}{d r}\right|^{2} \frac{d r}{\sqrt{\frac{2}{m}\left[V_{\min }-V(r)\right]}}
$$

The integral can be split into two integrals.

$$
\times\left[\int_{0}^{r_{m i n}}\left|\frac{d V}{d r}\right|^{2} \frac{d r}{\sqrt{\left[V_{\text {min }}-V(r)\right]}}+\int_{r_{\text {min }}}^{\infty}\left|\frac{d V}{d r}\right|^{2} \frac{d r}{3 c^{3} m^{2}} \sqrt{\frac{m}{2}} \sqrt{\sqrt{\left[V_{\text {min }}-V(r)\right]}}\right]
$$

The region for the first integral is excluded because the particle will never go there, thus, the first integral vanishes. We are left with

$$
\begin{equation*}
\Delta W=\frac{4}{3} \frac{Z^{2} q^{2}}{c^{3} m^{2}} \sqrt{\frac{m}{2}} \int_{r_{\min }}^{\infty}\left|\frac{d V}{d r}\right|^{2} \frac{d r}{\sqrt{\left[V_{\min }-V(r)\right]}} \tag{11}
\end{equation*}
$$

quod erat demonstrandum.
b. For the Coulomb potential, $V_{c}(r)=\frac{z Z q^{2}}{r}$, find the total energy radiated.
First, $\frac{d V_{c}}{d r}=-\frac{z Z q^{2}}{r^{2}}=-\frac{V_{c}}{r}$. Also, we can solve for $d r$.

$$
d V_{c}=-\frac{V_{c}}{r} d r \rightarrow d r=-\frac{r^{2} d V}{z Z q^{2}}
$$

Plug $V_{c}(r)$ and $d r$ into equation 11:

$$
\Delta W=-\frac{4}{3} \frac{Z^{2} q^{2}}{c^{3} m^{2}} \sqrt{\frac{m}{2}} \int_{a}^{0} \frac{V_{c}{ }^{2}}{r} \frac{\frac{r^{2}}{z Z q^{2}} d V_{c}}{\sqrt{\left[\frac{m v_{0}^{2}}{2}-V_{c}\right]}}=-\frac{4}{3} \frac{Z}{z m^{2} c^{3}} \sqrt{\frac{m}{2}} \int_{a}^{0} \frac{V_{c}^{2} d V_{c}}{\sqrt{a-V_{c}}}
$$

The limits of integration have been changed $V\left(r_{\text {min }}\right)=\frac{m v_{0}^{2}}{2}=a$ and $V(\infty)=$ 0.

The integral can be evaluated using your favorite table of integrals.

$$
\int \frac{x^{2} d x}{\sqrt{A-x}}=-\sqrt{A-x}\left(\frac{16 A^{2}}{15}+\frac{8 A x}{15}+\frac{2 x^{2}}{5}\right)
$$

So the integral equals $-\frac{16 a^{2}}{15} \sqrt{a}$, and finally, we have

$$
\Delta W=\frac{4}{3} \frac{Z}{z m^{2} c^{3}} \sqrt{\frac{m}{2}} \frac{16}{15}\left(\frac{m v_{0}^{2}}{2}\right)^{\frac{5}{2}}=\frac{8}{45} \frac{Z m v_{o}^{5}}{z c^{3}}
$$


[^0]:    ${ }^{1}$ excluding of course charge contained within any cavities

[^1]:    ${ }^{2}$ I've been a little redundant with the subscript and the prime, but I felt clarity was better than brevity at this point.

[^2]:    ${ }^{3}$ Solve the integral numerically.

[^3]:    ${ }^{4}$ For the more mathematically oriented, this should sound cacaphonous.

[^4]:    ${ }^{5}$ The use of rapidity is not my own clever innovation. My prof. suggested this.

[^5]:    ${ }^{6}$ In Arabic, al-jabra means roughly to compel, so this is really just a terrible pun

